POBABILITY AND STATISTICS, Lesson 2.
Conditional Probability, Independence

- *Definition.* $\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}, \quad \mathbb{P}(B) > 0.$

  With B fixed, $\mathbb{Q}(A) := \mathbb{P}(A|B)$. $(\Omega, \mathcal{A}, \mathbb{Q})$ is also a probability space with all of its consequences.

- *Definition.* $B_1, B_2, \dots$ is a complete set of mutually exclusive events, if $B_iB_j = \emptyset$ $(i \neq j)$ and $\sum_i \mathbb{P}(B_i) = 1$.

- **Theorem.** Let $B_1, B_2, \dots$ be a complete set of mutually exclusive events and A be an arbitrary event. Then

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i).$$

- **Theorem (Bayes).** Let $B_1, B_2, \dots$ be a complete set of mutually exclusive events and A be an arbitrary event. Then

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k)}{\sum_i \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i)}, \qquad k = 1, 2, \dots$$

- **Theorem** (factorization). Let $A_1, A_2, \dots, A_n$ be arbitrary events. Then

$$\mathbb{P}(A_1A_2 \dots A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \dots \mathbb{P}(A_n|A_1 \dots A_{n-1}).$$

- *Definition.* A and B are independent, if

$$\mathbb{P}(AB) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

  Remark: if $\mathbb{P}(A) \neq 0$ and $\mathbb{P}(B) \neq 0$, then A and B cannot be exclusive and independent at the same time. Ω and $\emptyset$ are independent of any other event.

- *Definition.* The events $A_1, \dots, A_n$ are (completely) independent, if

$$\mathbb{P}(A_{i_1} \dots A_{i_k}) = \mathbb{P}(A_{i_1}) \dots \mathbb{P}(A_{i_k})$$

  for any k-tuple $A_{i_1}, \dots, A_{i_k}$ and $k = 2, \dots, n$. ($k = 2$ case: pairwise independence, weaker than independence.)

- **Theorem (Borel–Cantelli Lemma)**

  1. Let $A_1, A_2, \dots$ be arbitrary events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Then

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = 0.$$

  2. Let $A_1, A_2, \dots$ be (completely) independent events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Then

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = 1.$$

- **Theorem (Lovász Local Lemma).** Let $A_1, A_2, \dots$ be events with *dependency graph* $G = (V, E)$, where $V = \{v_1, \dots, v_n\}$ corresponds to the events and A_i is independent of $\{A_j \,|\, \{v_i, v_j\} \notin E\}$. Suppose that

$$\mathbb{P}(A_i) \leq \frac{1}{4d_{\max}(G)} \qquad (i = 1, \dots, n),$$

  where $d_{\max}(G)$ is the maximum vertex degree in G. Then

$$\mathbb{P}(\overline{A}_1 \dots \overline{A}_n) > 0.$$