POBABILITY AND STATISTICS, Lesson 6.

- **The Distribution of Transformed Random Variables**

1. Let $t : X \to Y$ be invertible trf. The p.d.f. of X is $f(x)$, the p.d.f. of Y is $g(y)$:

$$g(y) = f(t^{-1}(y)) \cdot |\frac{d}{dy}t^{-1}(y)|, \qquad y = t(x) \quad \text{for some } x \in \text{supp}(f).$$

2. Let $t : \mathbf{X} \to \mathbf{Y}$ be invertible trf. The p.d.f. of $\mathbf{X}$ is $f(\mathbf{x})$, the p.d.f. of $\mathbf{Y}$ is $g(\mathbf{y})$:

$$g(\mathbf{y}) = f(t^{-1}(\mathbf{y})) \cdot |\det(\frac{\partial \mathbf{x}}{\partial \mathbf{y}})|, \qquad \mathbf{y} = t(\mathbf{x}) \quad \text{for some } \mathbf{x} \in \text{supp}(f).$$

3. Let $t : (X, Y) \to Z$ be trf. The joint p.d.f. of (X, Y) is $f(x, y)$, the c.d.f. of Z is $H(z)$:

$$H(z) = \iint_{\{(x,y)|t(x,y)<z\}} f(x, y)\, dxdy, \quad z = t(x, y) \quad \text{for some (usually many) } (x, y) \in \text{supp}(f).$$

4. *Convolution.* Let X and Y be independent r.v.'s with p.d.f. $f_1(x)$ and $f_2(y)$, respectively. The p.d.f. of $Z = X + Y$:

$$h(z) = \int_{-\infty}^{\infty} f_1(x) \cdot f_2(z - x)\, dx = \int_{-\infty}^{\infty} f_2(y) \cdot f_1(z - y)\, dy.$$

- **Useful Inequalities**

1. *Markov's Inequality:* Let X be a r.v. of finite first moment and taking on nonnegative values. Then

$$\mathbb{P}(X \geq c) \leq \frac{\mathbb{E}(X)}{c}, \qquad \forall c > 0.$$

2. *Chebyshev's Inequality:* Let X be a r.v. with finite second moment. Then

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \varepsilon) \leq \frac{\mathbb{D}^2(X)}{\varepsilon^2}, \qquad \forall \varepsilon > 0.$$

3. *Chernoff's Inequality:* $X_1, \ldots, X_n$ i.i.d., $|X_i| \leq K$, $X := \sum_{i=1}^n X_i$. Then

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a) \leq e^{-\frac{a^2}{2(\mathbb{D}^2(X)+Ka/3)}}, \qquad \forall a > 0.$$

- **Laws of Large Numbers, Central Limit Theorem**

1. *Weak Law*: If $X_1, \ldots, X_n$ are i.i.d. with finite $\mathbb{E}(X_i) = \mu$, then $\overline{X}_n \to \mu$ in probability:

$$\lim_{n\to\infty} \mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) = 0, \quad \forall \varepsilon > 0, \text{ where } \overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i.$$

2. *Strong Law*: If $X_1, \ldots, X_n$ are i.i.d. with finite $\mathbb{E}(X_i) = \mu$, then $\overline{X}_n \to \mu$ almost surely:

$$\mathbb{P}(\lim_{n\to\infty} \overline{X}_n = \mu) = 1.$$

3. **CLT**: If $X_1, \ldots, X_n$ are i.i.d. with finite $\mathbb{E}(X_i) = \mu$ and $\mathbb{D}^2(X_i) = \sigma^2$, then

$$\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} \Rightarrow \mathcal{N}(0, 1) \quad \text{in distribution (convergence of c.d.f.'s)}, \quad n \to \infty.$$