DYNAMIC FACTOR ANALYSIS

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Motivation

- Having multivariate time series, e.g., financial or economic data observed at regular time intervals, we want to describe the components of the time series with a smaller number of uncorrelated factors.
- The usual factor model of multivariate analysis cannot be applied immediately as the factor process also varies in time.
- There is a dynamic part, added to the usual factor model, the auto-regressive process of the factors.
- Dynamic factors can be identified with some latent driving forces of the whole process. Factors can be identified only by the expert (e.g., monetary factors) .

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Remarks

- The model is applicable to weakly stationary (covariance-stationary) multivariate processes.
- The first descriptions of the model is found in J. F. Geweke, International Economic Review 22 (1977) and in Gy. Bánkövi et. al., Zeitschrift für Angewandte Mathematik und Mechanik 63 (1981).
- Since then, the model has been developed in such a way that dynamic factors can be extracted not only sequentially, but at the same time. For tis purpose we had to solve the problem of finding extrema of inhomogeneous quadratic forms in Bolla et. al., Lin. Alg. Appl. 269 (1998).

The model

The input data are n-dimensional observations $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))$, where t is the time and the process is observed at discrete moments between two limits $(t = t_1, \ldots, t_2)$. For given positive integer $M < n$ we are looking for uncorrelated factors $F_1(t), \ldots, F_M(t)$ such that they satisfy the following model equations:

1. As in the usual linear model,

$$F_m(t) = \sum_{i=1}^{n} b_{mi} y_i(t), \quad t = t_1, \ldots, t_2;\ m = 1, \ldots, M. \qquad (1)$$

2. The dynamic equation of the factors:

$$\hat{F}_m(t) = c_{m0} + \sum_{k=1}^{L} c_{mk} F_m(t-k), \quad t = t_1 + L, \ldots, t_2;\ m = 1, \ldots, M, \tag{2}$$

where the time-lag L is a given positive integer and $\hat{F}_m(t)$ is the auto-regressive prediction of the mth factor at date t (the white-noise term is omitted, therefore we use $\hat{F}_m$ instead of F_m).

3. The linear prediction of the variables by the factors as in the usual factor model:

$$\hat{y}_i(t) = d_{0i} + \sum_{m=1}^{M} d_{mi} F_m(t), \quad t = t_1, \dots, t_2;\ i = 1, \dots, n. \tag{3}$$

(The error term is also omitted, that is why we use the notation $\hat{y}_i$ instead of y_i.)

The objective function

We want to estimate the parameters of the model:
$\mathbf{B} = (b_{mi})$, $\mathbf{C} = (c_{mk})$, $\mathbf{D} = (d_{mi})$
$(m = 1, \dots, M;\ i = 1, \dots, n;\ k = 1, \dots L)$
in matrix notation (estimates of the parameters c_{m0}, d_{0i} follow from these) such that the objective function

$$w_0 \cdot \sum_{m=1}^{M} \operatorname{var}(F_m - \hat{F}_m)_L + \sum_{i=1}^{n} w_i \cdot \operatorname{var}(y_i - \hat{y}_i) \tag{4}$$

is minimum on the conditions for the orthogonality and variance of the factors:

$$\operatorname{cov}(F_m, F_l) = 0, \quad m \neq l; \quad \operatorname{var}(F_m) = v_m, \quad m = 1, \dots, M \tag{5}$$

where $w_0, w_1, \dots, w_n$ are given non-negative constants (balancing between the dynamic and static part), while the positive numbers v_m's indicate the relative importance of the individual factors.

Notation

In Bánkövi et al., authors use the same weights

$$v_m = t_2 - t_1 + 1, \qquad m = 1, \ldots, M.$$

Denote

$$\bar{y}_i = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} y_i(t)$$

the sample mean (average with respect to the time) of the ith component,

$$\operatorname{cov}(y_i, y_j) = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} (y_i(t) - \bar{y}_i) \cdot (y_j(t) - \bar{y}_j)$$

the sample covariance between the ith and jth components, while

$$\operatorname{cov}^*(y_i, y_j) = \frac{1}{t_2 - t_1} \sum_{t=t_1}^{t_2} (y_i(t) - \bar{y}_i) \cdot (y_j(t) - \bar{y}_j)$$

the corrected empirical covariance between them.

Estimating the model parameters

The parameters c_{m0}, d_{0i} can be written in terms of the other parameters:

$$c_{m0} = \frac{1}{t_2 - t_1 - L + 1} \sum_{t=t_1+L}^{t_2} \left(F_m(t) - \sum_{k=1}^{L} c_{mk} F_m(t-k)\right),$$

$$m = 1, \ldots, M$$

and

$$d_{0i} = \bar{y}_i - \sum_{m=1}^{M} d_{mi} \bar{F}_m,$$

$$i = 1, \ldots, n.$$

Further notation

Thus, the parameters to be estimated are collected in the $M \times n$ matrices $\mathbf{B}, \mathbf{D}$, and in the $M \times L$ matrix $\mathbf{C}$.
$\mathbf{b}_m \in \mathbb{R}^n$ be the mth row of matrix $\mathbf{B}$, $m = 1, \dots, M$.

$$Y_{ij} := \operatorname{cov}(y_i, y_j), \qquad i, j = 1, \dots n,$$

and $\mathbf{Y} := (Y_{ij})$ is the $n \times n$ symmetric, positive semidefinite empirical covariance matrix of the sample (sometimes it is corrected).

The delayed time series:

$$z_i^m(t) = y_i(t) - \sum_{k=1}^{L} c_{mk} y_i(t-k), \tag{6}$$

$$t = t_1 + L, \ldots, t_2; \quad i = 1, \ldots, n; \quad m = 1, \ldots, M$$

and

$$Z_{ij}^m := \operatorname{cov}\left(z_i^m, z_j^m\right) =$$

$$= \frac{1}{t_2 - t_1 - L + 1} \sum_{t=t_1+L}^{t_2} (z_i^m(t) - \bar{z}_i^m) \cdot (z_j^m(t) - \bar{z}_j^m), \tag{7}$$

$$i, j = 1, \ldots n,$$

where $\bar{z}_i^m = \frac{1}{t_2 - t_1 - L + 1} \sum_{t=t_1+L}^{t_2} z_i^m(t)$, $i = 1, \ldots, n$; $m = 1, \ldots, M$.

The objective function revisited

Let $\mathbf{Z}^m = (Z_{ij}^m)$ be the $n \times n$ symmetric, positive semidefinite covariance matrix of these variables.
The objective function of (4) to be minimized:

$$G(\mathbf{B}, \mathbf{C}, \mathbf{D}) = w_0 \sum_{m=1}^{M} \mathbf{b}_m^T \mathbf{Z}^m \mathbf{b}_m + \sum_{i=1}^{n} w_i Y_{ii} -$$

$$-2 \sum_{i=1}^{n} w_i \sum_{m=1}^{M} d_{mi} \sum_{j=1}^{n} b_{mj} Y_{ij} + \sum_{i=1}^{n} w_i \sum_{m=1}^{M} d_{mi}^2 v_m,$$

where the minimum is taken on the constraints

$$\mathbf{b}_m^T \mathbf{Y} \mathbf{b}_l = \delta_{ml} \cdot v_m, \quad m, l = 1, \dots, M. \tag{8}$$

Outer cycle of the iteration

Choosing an initial $\mathbf{B}$ satisfying (8), the following two steps are alternated:

1. Starting with $\mathbf{B}$ we calculate the F_m's based on (1), then we fit a linear model to estimate the parameters of the autoregressive model (2). Hence, the current value of $\mathbf{C}$ is obtained.
2. Based on this $\mathbf{C}$, we find matrices $\mathbf{Z}^m$ using (6) and (7) (actually, to obtain $\mathbf{Z}^m$, the mth row of $\mathbf{C}$ is needed only), $m = 1, \dots, M$. Putting it into $G(\mathbf{B}, \mathbf{C}, \mathbf{D})$, we take its minimum with respect to $\mathbf{B}$ and $\mathbf{D}$, while keeping $\mathbf{C}$ fixed.

With this $\mathbf{B}$, we return to the 1st step of the outer cycle and proceed until convergence.

Fixing $\mathbf{C}$, the part of the objective function to be minimized in $\mathbf{B}$ and $\mathbf{D}$ is

$$F(\mathbf{B},\mathbf{D}) = w_0 \sum_{m=1}^{M} \mathbf{b}_m^T \mathbf{Z}^m \mathbf{b}_m + \sum_{i=1}^{n} w_i \sum_{m=1}^{M} d_{mi}^2 v_m -$$

$$-2 \sum_{i=1}^{n} w_i \sum_{m=1}^{M} d_{mi} \sum_{j=1}^{n} b_{mj} Y_{ij},$$

Taking the derivative with respect to $\mathbf{D}$:

$$F(\mathbf{B},\mathbf{D}^{opt}) = w_0 \sum_{m=1}^{M} \mathbf{b}_m^T \mathbf{Z}^m \mathbf{b}_m - \sum_{i=1}^{n} w_i \sum_{m=1}^{M} \frac{1}{v_m} (\sum_{j=1}^{n} b_{mj} Y_{ij})^2.$$

Introducing $V_{jk} = \sum_{i=1}^{n} w_i Y_{ij} Y_{ik}$, $\mathbf{V} = (V_{jk})$, and

$$\mathbf{S}_m = w_0 \mathbf{Z}^m - \frac{1}{v_m}\mathbf{V}, \quad m = 1, \ldots, M$$

we have

$$F(\mathbf{B},\mathbf{D}^{opt}) = \sum_{m=1}^{M} \mathbf{b}_m^T \mathbf{S}_m \mathbf{b}_m. \tag{9}$$

Thus, $F(\mathbf{B}, \mathbf{D}^{opt})$ is to be minimized on the constraints for $\mathbf{b}_m$'s. Transforming the vectors $\mathbf{b}_1, \dots, \mathbf{b}_m$ into an orthonormal set, an algorithm to find extrema of inhomogeneous quadratic forms is to be used.

The transformation

$$\mathbf{x}_m := \frac{1}{\sqrt{v_m}} \mathbf{Y}^{1/2} \mathbf{b}_m, \quad \mathbf{A}_m := v_m \mathbf{Y}^{-1/2} \mathbf{S}_m \mathbf{Y}^{-1/2}, \; m = 1, \dots, M \tag{10}$$

will result in an orthonormal set $\mathbf{x}_1, \dots, \mathbf{x}_M \in \mathbb{R}^n$, further

$$F(\mathbf{B}, \mathbf{D}^{opt}) = \sum_{m=1}^{M} \mathbf{x}_m^T \mathbf{A}_m \mathbf{x}_m,$$

and by back transformation:

$$\mathbf{b}_m^{opt} = \sqrt{v_m} \mathbf{Y}^{-1/2} \mathbf{x}_m^{opt}, \quad m = 1, \dots, M.$$

Hungarian Republic, 1993–2007

VARIABLES OF THE MODEL

Gross Domestic Product (1000 million HUF) – GDP
Number of Students in Higher Education – EDU
Number of Hospital Beds – HEALTH
Industrial Production (1000 million HUF) – IND
Agricultural Area (1000 ha) – AGR
Energy Production (petajoule) – ENERGY
Energy Import (petajoule) – IMP
Energy Export (petajoule) – EXP
National Economic Investments (1000 million HUF) – INV
Number of Scientific Publications – PUBL

Figure

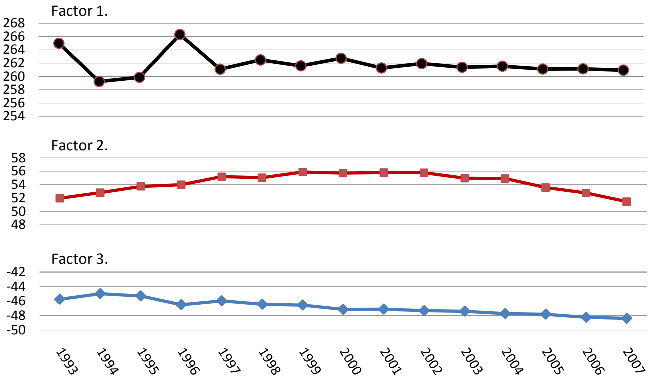

Figure 1. The Factor Process

bciprus2.txt

Time	factor 1.	factor 2.	factor 3.
1993	264.977	51.972	-45.760
1994	259.219	52.811	-44.986
1995	259.846	53.737	-45.308
1996	266.300	53.996	-46.514
1997	261.073	55.183	-45.988
1998	262.468	55.033	-46.456
1999	261.569	55.879	-46.562
2000	262.729	55.729	-47.168
2001	261.258	55.788	-47.138
2002	261.933	55.781	-47.337
2003	261.361	54.962	-47.418
2004	261.529	54.896	-47.736
2005	261.107	53.557	-47.833
2006	261.118	52.758	-48.254
2007	260.925	51.465	-48.401

Table: Estimation of the Factors

bciprus3.txt

	factor 1.	factor 2.	factor 3.
GDP	38.324	-2.541	-6.116
EDU	-1.775	5.725	0.015
HEALTH	10.166	0.837	-1.650
IND	-0.261	0.255	-0.107
AGR	6.146	2.919	-1.124
ENERGY	24.082	4.592	-4.054
IMP	1.560	-1.209	-0.213
EXP	-3.907	-0.233	0.615
INV	2.864	0.038	-0.510
PUBL	-0.608	0.197	0.089

Table: Factor Loadings (matrix $\mathbf{B}$)

bciprus4.txt

	factor 1.	factor 2.	factor 3.	Constant term
GDP	-0.108	-0.025	-0.677	-0.670
EDU	-0.142	0.145	-0.877	-8.637
HEALTH	0.115	-0.132	0.656	16.250
IND	-0.898	-0.187	-5.784	-14.690
AGR	0.021	0.005	0.137	6.809
ENERGY	0.085	-0.038	0.543	10.055
IMP	-0.098	-0.152	-0.868	0.311
EXP	-0.516	-0.931	-1.840	109.915
INV	-0.209	0.026	-1.341	-6.779
PUBL	-0.061	0.121	-0.484	-9.867

Table: Variables Estimated by The Factors (matrix **D**)

bciprus5.txt

Timelag	factor 1.	factor 2.	factor 3.
0	-0.000	0.001	-0.000
1	0.069	0.283	0.117
2	0.473	1.644	0.495
3	0.205	0.229	0.141
4	0.251	-1.168	0.258

Table: Dynamic Equations of The Factors (matrix **C**)

Extrema of sums of inhomogeneous quadratic forms

Given the $n \times n$ symmetric matrices $\mathbf{A}_1, \dots, \mathbf{A}_k$ ($k \leq n$) we are looking for an orthonormal set of vectors $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A}_i \mathbf{x}_i \to \text{maximum}.$$

Theoretical solution

By Lagrange's multipliers the $\mathbf{x}_i$'s giving the optimum satisfy the system of linear equations

$$A(\mathbf{X}) = \mathbf{XS} \tag{11}$$

with some $k \times k$ symmetric matrix $\mathbf{S}$, where the $n \times k$ matrices $\mathbf{X}$ and $A(\mathbf{X})$ are as follows:

$$\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_k), \quad A(\mathbf{X}) = (\mathbf{A}_1\mathbf{x}_1, \dots, \mathbf{A}_k\mathbf{x}_k).$$

Due to the constraints imposed on $\mathbf{x}_1, \dots, \mathbf{x}_k$, the non-linear system of equations

$$\mathbf{X}^T\mathbf{X} = \mathbf{I}_k \tag{12}$$

must also hold.

As $\mathbf{X}$ and the symmetric matrix $\mathbf{S}$ contain alltogether $nk + k(k+1)/2$ free parameters, while the equations (11) and (12) contain the same number of equations, the solution of the problem is expected. Transform (11) into a homogeneous system of linear equations, to get a non-trivial solution,

$$|\mathbf{A} - \mathbf{I}_n \otimes \mathbf{S}| = 0 \qquad (13)$$

must hold, where the $nk \times nk$ matrix $\mathbf{A}$ is a Kronecker-sum $\mathbf{A} = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_k$ ($\otimes$ denotes the Kronecker-product).
Generalization of the eigenvalue problem: eigenmatrix problem.

Numerical solution

Starting with a matrix $\mathbf{X}^{(0)}$ of orthonormal columns, the mth step of the iteration is as follows ($m = 1, 2, \dots$):
Take the polar decomposition

$$A(\mathbf{X}^{(m-1)}) = \mathbf{X}^{(m)} \cdot \mathbf{S}^{(m)}$$

into an $n \times k$ suborthogonal matrix (a matrix of orthonormal columns) and a $k \times k$ symmetric matrix ($k \leq n$). Let the first factor be $\mathbf{X}^{(m)}$, etc. until convergence. In fact, the trace of $\mathbf{S}^{(m)}$ converges to the optimum of the objective function.
The polar decomposition is obtained by SVD.
The above iteration is easily adopted to negative semidefinite or indefinite matrices and to finding minima instead of maxima.

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