

# High-Dimensional Dynamic Factor Models: A Selective Survey and Lines of Future Research

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## Abstract

High-Dimensional Dynamic Factor Models are presented in detail: The main assumptions and their motivation, main results, illustrations by means of elementary examples. In particular, the role of singular ARMA models in the theory and applications of High-Dimensional Dynamic Factor Models is discussed. The emphasis of the paper is on model classes and their structure theory, rather than on estimation in the narrow sense. Our aim is not a comprehensive survey. Rather we try to point out promising lines of research and applications that have not yet been sufficiently developed.

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# 1 Introduction

Analysis and forecasting of high-dimensional time series recently has attracted substantial interest, see Hallin et al. (2020). However, “classical” multivariate time-series models such as autoregressive models suffer from the so-called “curse of dimensionality”: Unless additional restrictions are imposed, the parameter spaces grow with the square of the dimension of the time series,  $N$  say; thus in many cases, even for moderate  $N$ , the sample size available is not sufficient to guarantee reliable parameter estimation. This is true in particular for macroeconomic applications. In this case, on the one hand, the interaction of several variables, disaggregated series in particular, may be important for modeling and forecasting. On the other hand, sample size of macroeconomic time series is in many cases rather limited, e.g. because of structural changes in the underlying economies.

High-dimensional dynamic factor models are one way to overcome this curse of dimensionality. Factor models for the case of i.i.d. observations have a long history, dating back to Spearman (1904) and Burt (1909). Factor models in time-series context have been proposed much later, in particular by Geweke (1977) and Sargent and Sims (1977), Watson and Engle (1983), Quah and Sargent (1993). The basic idea is to separately model the comovement between the variables on the one hand, and the individual, or idiosyncratic movement on the other hand. Equivalently, the variables are decomposed as the sum of a *latent variable*, or *common component*, and an *idiosyncratic* component.

The above factor models are “exact” in the sense that the idiosyncratic components are assumed to be cross-sectionally uncorrelated at all leads and lags. Inspired by the idea of risk diversification, Chamberlain (1983) and Chamberlain and Rothschild (1983) introduced the notion of “approximate” factor models, which are high-dimensional (potentially infinite-dimensional) models where the idiosyncratic terms are allowed to be cross-sectionally dependent, although in a weak sense. This idea has been extended to the linear dynamic case by Forni et al. (2000), Forni and Lippi (2001), Stock and Watson (2002a,b), Bai and Ng (2002), Bai (2003), leading to the class of linear High-Dimensional Dynamic Factor Models, Dynamic Factor Models for short (DFM).

The present paper does not contain a comprehensive survey. We go over the main assumptions and results with the purpose of pointing out lines of research that have not yet been sufficiently developed.

In Section 2 we describe the model class of DFMs, thus the decomposition of the observable  $N$ -dimensional vector  $y_t^N$  into common and idiosyncratic components,  $\chi_t^N$  and  $\xi_t^N$  respectively. We assume that  $\chi_t^N$  has rational spectral density  $f_\chi^N$ . The rank  $q$  of  $f_\chi^N$  does not depend on  $N$ , for  $N$  sufficiently large. The state dimension

of minimal, stable and miniphase state space realization of  $f_x^N$  is independent of  $N$ . (This can be rephrased as saying, that there is a finite number of “static” factors, see Section 3.)

In Section 3 we discuss the model for the process of the common components. Precisely, the common components are represented as linear combinations of a finite-dimensional vector process, whose coordinates are called the (minimal) static factors. The latter are modeled as an ARMA system driven by a vector white noise whose coordinates are called the dynamic factors. Here we introduce the so-called singular ARMA (or state-space) systems. These generate  $N$ -dimensional stationary processes with rational spectral density, whose rank is less than  $N$ : we call them singular<sup>1</sup>. We argue that, under our assumptions, this is obviously the case for the common-component vector, and is very likely for the factor vector. On the other hand, it has been shown that singular ARMA processes can (generically) be modeled as finite AR’s, see Section 3.3, which implies a most important simplification in the modeling of common components and static factors.

In Section 4 we describe techniques to obtain the common components and factors from the observable vector  $y_t^N$ , for  $N$  and  $T$  (the number of observations for each time series) tending to infinity. In Section 4.1 we show how principal components (PCA) can be used. The underlying estimation procedure is to estimate the static factors by PCA in a first step and then to estimate an ARMA or AR model in order to describe the dynamics of the static factors. An alternative approach, see Section 4.3, is to assume a dynamic factor model with autoregressive static factors and cross-sectionally uncorrelated idiosyncratic components, thus an exact factor model, and to put this in a state-space framework. In this framework (once the integer specification parameters have been fixed) an EM algorithm with the E-step based on Kalman filtering can be applied. That an exact factor model can be used to estimate a DFM has been shown in Doz et al. (2012), see 4.3.

In the case where no  $N$ -independent static factors exist, frequency-domain methods, as described in Section 4.2, may be applied.

In Section 5.1 we present some applications of the results on singular ARMA models to empirical macroeconomic analysis, to the so-called fundamentalness problem in particular. Some results on cointegration for singular ARIMAs are presented in Section 5.2.

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<sup>1</sup>Not to be confused with singular processes in the sense of Kolmogorov.

## 2 High-Dimensional Dynamic Factor Models (DFM). The Model Class

The basic idea is to represent the  $N$ -dimensional observation vector at time  $t \in \mathbb{Z}$ ,  $y_t^N$  say, as

$$y_t^N = \chi_t^N + \xi_t^N, \quad (2.1)$$

where  $(\chi_t^N \mid t \in \mathbb{Z})$  and  $(\xi_t^N \mid t \in \mathbb{Z})$  are the  $N$ -dimensional processes of common and idiosyncratic components respectively. The one-dimensional processes  $(\chi_{it}, t \in \mathbb{Z})$  are strongly dependent across the index  $i$ , whereas the processes  $(\xi_{it}, t \in \mathbb{Z})$  are weakly dependent. The precise meaning of strong and weak dependence is given below.

Throughout, except for Section 5.2, we assume that  $(\chi_t^N)$  and  $(\xi_t^N)$  are wide-sense stationary. In addition, throughout we assume

$$\mathbb{E}\chi_t^N = \mathbb{E}\xi_t^N = 0 \quad \forall t \quad (2.2)$$

$$\mathbb{E}\chi_t^N \xi_s^{N'} = 0 \quad \forall t, s, \quad (2.3)$$

and that the spectral densities of  $(\chi_t^N)$  and  $(\xi_t^N)$  exist. As a consequence,  $(y_t^N)$  is stationary and has a spectral density, which, using an obvious notation, is:

$$f_y^N(\lambda) = f_\chi^N(\lambda) + f_\xi^N(\lambda), \quad \lambda \in [-\pi, \pi]. \quad (2.4)$$

Throughout,  $z$  is used for a complex variable as well as for the backward shift on  $\mathbb{Z}$ .

The following assumptions constitute the class of DFM's considered in the present paper (we follow here Deistler et al. (2010)):

**Assumption 1.** *For all  $N$ ,  $f_\chi^N$  is a rational spectral density.*

An obvious consequence of Assumption 1 is that  $f_\chi^N$  has constant normal rank, i.e. has the same rank almost everywhere on  $[-\pi, \pi]$ .

Here, for asymptotic analysis not only the sample size  $T$ , but also the cross-sectional dimension  $N$  is tending to infinity—this has an empirical motivation in the study of high-dimensional vector time series, i.e. vector time series whose dimension is allowed to be close to or even higher than the sample size. Thus the underlying process considered is a double-indexed stochastic process  $(y_{it} \mid i \in \mathbb{N}, t \in \mathbb{Z})$ , corresponding, as  $N$  varies, to a nested sequence of models (2.1), in the sense that  $y_{it}$ ,  $\chi_{it}$ , and  $\xi_{it}$  do not depend on  $N$ , for  $i \leq N$ .

**Assumption 2.** *We suppose that there exists  $N_0 \geq q$  such that, from  $N_0$  onwards, the rank of  $f_\chi^N$  is independent of  $N$ . Such rank is denoted by  $q$ .*

As is well known, see e.g. Hannan and Deistler (2012), every rational spectral density can be described by an ARMA or alternatively a state space system.

**Assumption 3.** *The state dimension of a minimal, stable and miniphase state space realization corresponding to  $f_\chi^N$  is independent of  $N$  from a certain  $N_1$  onwards. Such state dimension is denoted by  $n$ .*

Of course, without loss of generality, we can assume  $N_0 = N_1$ . Next we define weak and strong dependence. We use  $\omega_{\xi,s}^N(\lambda)$ ,  $\omega_{\chi,s}^N(\lambda)$ ,  $\omega_{y,s}^N(\lambda)$  to denote the  $s$ -th largest eigenvalue of the Hermitian matrix  $f_\xi^N(\lambda)$ ,  $f_\chi^N(\lambda)$ ,  $f_y^N(\lambda)$ , respectively.

**Assumption 4.** *Weak cross-sectional dependence of  $(\xi_t^N)$ . The eigenvalues  $\omega_{\xi,1}^N(\lambda)$  are uniformly bounded, i.e. there exists  $B > 0$  such that  $\omega_{\xi,1}^N(\lambda) < B$  for all  $N$  and  $\lambda$ .*

**Assumption 5.** *Strong cross-sectional dependence of  $(\chi_t^N)$ . The eigenvalues  $\omega_{\chi,s}^N(\lambda)$ ,  $s = 1, \dots, q$ , diverge as  $N \rightarrow \infty$ ,  $\lambda$  almost everywhere in  $[-\pi, \pi]$ .*

Note that Assumptions 4 and 5 place restrictions on the *cross-dependence* of the idiosyncratic and common component respectively, not on their autocorrelations. Note also that the assumptions defining the common and idiosyncratic terms are asymptotic, for the number  $N$  of observable variables tending to infinity, so that in empirical application the observable vector  $y_t^N$  is supposed to be high dimensional.

Forni and Lippi (2001) prove that representation (2.1) is identified. More precisely, if  $\tilde{\chi}_t^N$ ,  $\tilde{\xi}_t^N$  and the integer  $\tilde{q}$  fulfill

$$y_t^N = \tilde{\chi}_t^N + \tilde{\xi}_t^N, \quad \text{for all } N,$$

(2.2), (2.3) and Assumptions 1, 2, 4 and 5, then  $\tilde{q} = q$ ,  $\tilde{\chi}_t^N = \chi_t^N$ ,  $\tilde{\xi}_t^N = \xi_t^N$ .

DFM's generalize exact dynamic factor models, considered in Geweke (1977), Sargent and Sims (1977), where  $N$  is fixed and  $f_\xi^N(\lambda)$  is diagonal, i.e. the one-dimensional idiosyncratic processes are mutually uncorrelated at any lead and lag. Exact dynamic factor models are rather restrictive, in that the class of spectral densities corresponding to them is restricted, in particular if  $q$  is small relative to  $N$  (which is the most important case), see Scherrer and Deistler (1998). On the other hand, for  $q$  small enough in relation to  $N$ , identifiability results are available for fixed  $N$ , see again Scherrer and Deistler (1998).

In general, several interpretations for DFM's are possible. E.g., the components  $\xi_t^N$  may be interpreted as “measurement” error and  $\chi_t^N$  as unobserved “true” variables; this is in line with errors-in-variables models, which, from an abstract point of view, are the same as factor models. An alternative interpretation of (2.1) is the decomposition of observations into a part  $(\chi_t^N)$  representing the comovements between

the variables (for instance representing the “market” effect on all stock prices) and individual movements ( $\xi_t^N$ ) (representing the firm’s specific effect). The components  $\xi_{it}$  are usually called the *idiosyncratic components*, while the latent variables  $\chi_{it}$  are referred to as the *common components*.

It is easy to see that, given a bijective map  $g : \mathbb{N} \rightarrow \mathbb{N}$ , if Assumption 5 holds for the process  $(\chi_{it} \mid i \in \mathbb{N}, t \in \mathbb{Z})$ , then it holds for  $(\chi_{g(i),t} \mid i \in \mathbb{N}, t \in \mathbb{Z})$ , because the vector  $\chi_t^N$  is nested in the vector  $\chi_t^{g,M} = (\chi_{g(1),t} \cdots \chi_{g(M),t})$  for some  $M$ . As a consequence,  $\omega_{\chi,s}^N(\lambda) \leq \omega_{\chi,s}^{g,M}(\lambda)$ , see e.g. Forni and Lippi (2001), Fact M, (b), p. 1121. Thus Assumption 5 holds irrespective of the order of the variables  $y_{it}$  (and thus  $\chi_{it}$ ).

However, the speed of divergence of the eigenvalues  $\omega_{\chi,s}^N(\lambda)$ ,  $s = 1, \dots, q$ , depends on that order. A simple example is the following. Let  $(a_s, s \in \mathbb{N})$  be a square summable sequence of real numbers and  $(j_s, s \in \mathbb{N})$  a sequence of positive integers. Then let

$$\chi_{it} = b_i v_t,$$

where  $v_t$  is a unit-variance scalar white noise and the coefficients  $b_i$  are the following:

$$1, a_1, \dots, a_{j_1}, 1, a_{j_1+1}, \dots, a_{j_1+j_2}, 1, a_{j_1+j_2+1}, \dots \quad (2.5)$$

In this case we have

$$\omega_{\chi,1}^N(\lambda) = \sum_{h=1}^N b_h^2.$$

In particular, if  $N = K + j_1 + \dots + j_K$ ,

$$\omega_{\chi,1}^N(\lambda) = K + a_1^2 + a_2^2 + \dots + a_{j_1+\dots+j_K}^2.$$

Assuming that  $j_s = 1$  for all  $s$ , we see that, for  $N$  odd,

$$\lim_{N \rightarrow \infty} \frac{\omega_{\chi,1}^N(\lambda)}{N} = \lim_{N \rightarrow \infty} \frac{K + a_1^2 + \dots + a_K^2}{2K} = \frac{1}{2}.$$

On the other hand, assuming that  $j_s = s$ , we obtain a reordering of the coefficients  $b_i$ , with ones appearing at linearly increasing intervals. It is easy to see that in this case the eigenvalue  $\omega_{\chi,1}^N(\lambda)$  grows asymptotically at speed  $N^{1/2}$ . DFM’s in which the eigenvalues are assumed to diverge with rates  $N^\alpha$ ,  $0 < \alpha < 1$ , known as models with “weak factors”, necessarily rely on a particular assumption on the order of the variables, see Onatski (2012) and related literature.

## 3 Modeling the Latent Process

### 3.1 Singular ARMA and state space systems

By Assumptions 1 and 2, the latent process  $(\chi_t^N)$  has a rational and, for  $N > N_0 \geq q$ , singular spectral density. By its rationality, it can be represented by a singular ARMA or state space system, i.e. an ARMA or state space system with a singular innovation variance. For regular ARMA or state space systems, i.e. systems whose output has an a.e. nonsingular spectral density, we refer to Hannan and Deistler (2012), Deistler and Scherrer (2019); for the singular case see Section 3.3.

Hereafter, unless strictly necessary, we omit the superscript  $N$ . For the ARMA system, assuming that  $N \geq N_0$ , we have

$$P(z)\chi_t = Q(z)v_t, \quad P(z) = P_0 - \sum_{j=1}^S P_j z^j, \quad Q(z) = \sum_{j=0}^{S'} Q_j z^j, \quad (3.1)$$

where  $v_t$  is an orthonormal  $q$ -dimensional white noise, with  $P_j \in \mathbb{R}^{N \times N}$ ,  $Q_j \in \mathbb{R}^{N \times q}$ . Moreover,

$$\det P(z) \neq 0 \quad \text{for } |z| \leq 1, \quad (3.2)$$

which is the stability condition, and

$$\text{rank } Q(z) = q, \quad \text{for } |z| \leq 1, \quad (3.3)$$

which is the strict miniphase condition.

Thus the steady state solution of (3.1) is

$$\chi_t = P^{-1}(z)Q(z)v_t = k(z)v_t = \sum_{j=0}^{\infty} k_j v_{t-j}. \quad (3.4)$$

Due to the stability and the miniphase conditions, (3.4) is a Wold representation,  $v_t$  are innovations and the one-step ahead prediction error for  $\chi_t$ , given its past  $\chi_s$ ,  $s < t$ , is  $k_0 v_t$ , see below. The spectral density of  $(\chi_t)$  is of the form

$$f_\chi(\lambda) = (2\pi)^{-1} k(e^{-i\lambda}) k(e^{-i\lambda})^* \quad (3.5)$$

where  $*$  denotes the Hermite conjugation.  $f_\chi(\lambda)$  has rank  $q$  for all  $\lambda \in [-\pi, \pi]$ . Given  $f_\chi$ , under conditions (3.2) and (3.3),  $k(z)$  is unique up to postmultiplication by (constant) orthogonal matrices. Moreover:

**Assumption 6.** *We suppose that  $P_0 = I_N$  and that  $P(z)$  and  $Q(z)$  are left coprime (see Hannan and Deistler, 2012, p.41), i.e. the matrix  $(P(z) \ Q(z))$  has rank  $N$  for all  $z \in \mathbb{C}$ .*

Alternatively, by Assumptions 2 and 3, for  $N \geq N_0$ , to  $f_\chi(\lambda)$  there corresponds a minimal state-space realization

$$x_{t+1} = Fx_t + Gw_{t+1} \quad (3.6)$$

$$\chi_t = Hx_t, \quad (3.7)$$

where  $x_t$  is an  $n$ -dimensional state,  $w_t$  is a  $q$ -dimensional orthonormal white noise, where  $n$  and  $q$  are independent of  $N$ ,  $F \in \mathbb{R}^{n \times n}$ ,  $G \in \mathbb{R}^{n \times q}$ ,  $H \in \mathbb{R}^{1 \times n}$  are parameter matrices. The stability condition is

$$\rho(F) < 1, \quad (3.8)$$

where  $\rho(F)$  denotes the spectral radius of  $F$ , and

$$\text{rank} \begin{pmatrix} I - Fz & -G \\ H & 0 \end{pmatrix} = n + q, \quad \text{for } |z| \geq 1 \quad (3.9)$$

is the strict miniphase condition.

Under conditions (3.8) and (3.9),  $w_t$  is an innovation for  $\chi_t$ . Moreover, setting

$$K(z) = H(z^{-1}I - F)^{-1}G,$$

we have

$$\chi_t = H(I - Fz)^{-1}Gw_t = K(z)w_t. \quad (3.10)$$

Under our assumptions, the innovations and the transfer function are unique up to premultiplication and postmultiplication by orthogonal matrices, respectively, so that we can assume with no loss of generality that  $w_t = v_t$  and  $K(z) = k(z)$ , where  $v_t$  and  $k(z)$  are defined in (3.1) and (3.4) respectively.

Representation (3.6)–(3.7) implies that, for  $N \geq N_0$ , the dimension of the space spanned by  $\chi_{1t}, \chi_{2t}, \dots, \chi_{Nt}$ , call it  $\mathcal{S}_t^N$ , for any given  $t$ , does not exceed  $n$ . Therefore there exists  $\tilde{N} \geq N_0$  and  $r \leq n$  such that for  $N \geq \tilde{N}$ ,

$$\dim(\mathcal{S}_t^N) = r. \quad (3.11)$$

With no loss of generality we can assume that  $\tilde{N} = N_0$ . By (3.11),  $\chi_{N_0+k,t}$ ,  $k > 0$ , is a linear combination of  $\chi_{1t}, \chi_{2t}, \dots, \chi_{N_0,t}$ , so that  $\chi_{N_0+k,t} = H_i x_t$ ,  $H_i \in \mathbb{R}^{1 \times n}$ , where  $x_t$  is the state vector of the minimal state-space realization for  $N = N_0$ . Thus, representation (3.6)–(3.7) holds for all  $N \geq N_0$  with the same  $x_t, w_t, F$  and  $G$ , and nested matrices  $H$ . Otherwise stated, (3.6)–(3.7) hold with (3.7) replaced by  $\chi_{it} = H_i x_t$ , where  $H_i$  is the  $i$ -row of the matrix  $H$ .



Note that there is a Hilbert space (in the Hilbert space  $L_2$  of square integrable random variables) construction of a state-space system, obtained by projecting all future values of the output process  $(\chi_t)$  on the Hilbert space spanned by its past (see e.g. Akaike 1974). Then, by the rationality of the spectral density, the space spanned by these projections is finite dimensional and every basis is a minimal state, see e.g. Hannan and Deistler (2012), Deistler and Scherrer (2018). We will call this the Kalman-Akaike realization of a state-space system.

The state space system (3.6)–(3.7) corresponds to a definition of past and future, respectively, as  $\chi_s$ ,  $s \leq t$  and  $\chi_s$ ,  $s \geq t$ . If we, however, define the past as  $\chi_s$ ,  $s < t$ , the Kalman-Akaike realization leads to a system of the form

$$\bar{x}_{t+1} = \bar{F}\bar{x}_t + \bar{G}v_t \quad (3.12)$$

$$\chi_t = \bar{H}\bar{x}_t + B_0v_t \quad (3.13)$$

and the corresponding representation of the transfer function is

$$\bar{H}(Iz^{-1} - \bar{F})^{-1}\bar{G} + B_0. \quad (3.14)$$

Note that now we impose a stability and a minimum phase condition analogous to (3.8) and (3.9) respectively, for  $(\bar{F}, \bar{G}, \bar{H}, B_0)$ . Moreover,  $\bar{H}\bar{x}_t$  is the one-step ahead forecast for  $\chi_t$  and  $B_0v_t$  the corresponding one-step ahead prediction error.

When  $N > N_0 \geq q$ , so that  $(\chi_t)$  is a singular stochastic process, i.e. a process with a singular spectral density, the left inverse of  $k(z)$  is not unique. To see this, consider the Smith-McMillan form (see e.g. Hannan and Deistler (2012))

$$k(z) = u(z)d(z)v(z), \quad (3.15)$$

where  $u(z)$  and  $v(z)$  are  $N \times N$  and  $q \times q$ , respectively, unimodular polynomial matrices (i.e. their determinants are non-zero constants) and

$$d(z) = \begin{pmatrix} \frac{\epsilon_1(z)}{\psi_1(z)} & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \frac{\epsilon_q(z)}{\psi_q(z)} \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where the matrix of zeros at the bottom is  $(N - q) \times q$ ,  $\epsilon_i$  and  $\psi_i$ , for  $i = 1, \dots, q$ , are relatively prime monic polynomials,  $\epsilon_i$  divides  $\epsilon_{i+1}$  and  $\psi_{i+1}$  divides  $\psi_i$ . Then a particular causal left inverse is given by

$$h^-(z) = v^{-1}(z)(d'(z)d(z))^{-1}d'(z)u^{-1}(z). \quad (3.16)$$

As is easily seen,  $h^-(z)$  has no poles or zeros for  $|z| \leq 1$  so that

$$v_t = h^-(z)\chi_t \tag{3.17}$$

is a causal relation and thus  $(v_t)$  are indeed innovations.

In order to uniquely determine  $k(z)$  corresponding to the Wold decomposition from  $f_\chi$  (compare (3.5)), in addition to stability and the miniphase assumption, we have to remove its non-uniqueness caused by post-multiplying by a  $q \times q$  orthogonal matrix. This leads to the following assumption guaranteeing uniqueness of  $k(z)$ :

**Assumption 7.** *For  $q = r$ , we assume that  $k(0) = I_N$ , i.e.  $Q_0 = I_N$  holds. For  $q < r$ , we assume that the top  $q \times q$  submatrix of  $k(0)$  is (1) non-singular (which is the case generically) and (2) lower triangular (with non-zero diagonal elements).*

### 3.2 Static and dynamic factors

In Section 3.1 we have argued that by Assumption 3, under a suitable choice of  $N_0$ , for  $N \geq N_0$  the dimension of the space  $\mathcal{S}_t^N$  spanned by  $\chi_{it}$ ,  $i = 1, \dots, N$  is  $r$ . Let  $(f_t)$  be an  $r$ -dimensional process such that

- (i)  $f_t$  forms a basis in  $\mathcal{S}_t^N$ ,
- (ii)  $f_t = S\chi_t$ , with  $S$  independent of  $t$ .

Of course (i) and (ii) imply that  $(f_t)$  is weakly stationary with a rational spectral density and that

$$\chi_t = Lf_t, \tag{3.18}$$

where  $L$  is an  $N \times r$  matrix independent of  $t$ . The vector  $f_t$  is called a vector of (*minimal*) *static factors* and  $L$  the corresponding loading matrix.

As is easy to see, a minimal static factor is unique up to premultiplication by a constant non-singular matrix  $\mathcal{T}$  and the factor loading matrix  $L$  is unique up to postmultiplication by  $\mathcal{T}^{-1}$ . By (3.7), the state  $x_t$  is a static factor, though in general not a minimal one.

In particular, let  $(f_t)$  be a process of static factors corresponding to  $N_0$ . Because  $\dim \mathcal{S}_t^N = \dim \mathcal{S}_t^{N_0}$  for all  $N > N_0$ ,  $f_t$  is a basis in  $\mathcal{S}_t^N$  for all  $N > N_0$ . Thus there exists a representation (3.18) in which the factors are independent of  $N$  and the matrix  $L$  corresponding to  $N$  is nested in the matrix  $L$  corresponding to  $M$ , for  $M > N$ . In other words, there exist an  $r$ -dimensional vector of factors  $f_t$  and a matrix  $L \in \mathbb{R}^{\infty \times r}$  such that  $\chi_{it} = L_i f_t$ , for all  $i \in \mathbb{N}$ , where  $L_i$  is the  $i$ -th row of  $L$ .

Of course, for estimation uniqueness of  $f_t$  and  $L$  is desirable. A common normalization is to assume that the top  $r \times r$  submatrix of  $L$ ,  $L_1$  say, is nonsingular (which is the case, generically) and then to impose  $L_1 = I_r$ . Clearly this corresponds to the selection of the first  $r$  elements of  $\chi_t$  as (minimal) static factors. In this case the on-

and above-diagonal entries of  $E f_t f_t'$  are additional free parameters. We will refer to this normalization as the *standard normalization*. Another common normalization is to assume that  $E f_t f_t' = I_r$ . Then  $L$  is unique up to multiplication by an orthogonal matrix, which is made unique by assuming appropriate  $Q - R$  decompositions for  $L_1$ .

A special explicit form of the static factors can be obtained in the following way. Let  $R$  be an  $N \times r$  matrix such that  $RR' = \gamma_\chi(0)$ , the expected value of  $\chi_t \chi_t'$ . As  $\gamma_\chi(0)$  has rank  $r$  for  $N \geq N_0$ , we have  $\text{rank } R = r$  for  $N \geq N_0$  as well. Now define

$$f_t = (R'R)^{-1}R'\chi_t \quad (3.19)$$

$$= (R'R)^{-1}R'k(z)v_t = w(z)v_t, \quad (3.20)$$

(see (3.4) for the first equality in (3.20)). Note that  $f_t$  is orthonormal and that  $L = R$ .

When  $R = P$ , where  $P$  has the first  $r$  normalized eigenvectors of  $\gamma_\chi(0)$  on the columns, the factors are the first  $r$  principal components of  $\chi_t$ . Note that if we take, for each  $N$ ,  $f_t$  as the first  $r$  principal components of  $\chi_t^N$ , the factors depend on  $N$  and the matrices  $L$  are not nested. The same occurs if the principal components are normalized, that is if  $R = \sqrt{\Lambda^{-1}}P$ , where  $\Lambda$  is the diagonal matrix with the first  $r$  eigenvalues of  $\gamma_\chi(0)$  on the diagonal.

Obviously, if the rank of the  $r \times r$  spectral density of  $(f_t)$  were less than  $q$ , then by (3.18) Assumption 2 would be violated, so that  $r \geq q$  must hold. The spectral density of  $(f_t)$  is then nonsingular or singular, for  $r = q$  or  $r > q$  respectively. Thus the dynamics of the static factor process  $(f_t)$  can be represented by a nonsingular or singular ARMA process of type (3.1), respectively. By (3.18) and (3.20), the Hilbert spaces spanned by  $\chi_t$  and  $f_t$  are the same and therefore,  $v_t$  is an innovation for  $f_t$  as well and  $k(z) = Lw(z)$  is the corresponding transfer function.

The innovations  $(v_t)$  in (3.4), and (3.20), are called the *(minimal) dynamic factors*. As has been stated already, under our assumptions, for given  $(f_t)$  they are unique up to premultiplication by a non-singular matrix or by an orthogonal matrix if we assume orthonormality of  $v_t$ .

Piecing together what we have seen here and in the previous section, under Assumption 3 the latent variables can be represented as follows:

$$\chi_{it} = L_i f_t \quad (3.21)$$

$$\alpha(z)f_t = \beta(z)v_t, \quad (3.22)$$

for  $i \in \mathbb{N}$ , where  $L_i$  is the  $i$ -th row of the matrix  $L \in \mathbb{R}^{\infty \times r}$  and

$$\alpha(z) = I_r - \sum_{j=1}^p A_j z^j, \quad A_j \in \mathbb{R}^{r \times r}, \quad \beta(z) = \sum_{j=0}^m B_j z^j, \quad B_j \in \mathbb{R}^{r \times q},$$

and where we assume that  $\alpha(z)$  satisfies the stability condition,  $\beta(z)$  satisfies the miniphase condition (that is, respectively, (3.2) with  $P(z)$  replaced by  $\alpha(z)$  and (3.3) with  $Q(z)$  replaced by  $\beta(z)$ ) and that  $\alpha(z)$  and  $\beta(z)$  are left coprime as  $P(z)$  and  $Q(z)$  in Assumption 6.

Lastly, let us point out that normally the literature on DFM's assumes, rather than our Assumption 3, the existence of representation (3.21)–(3.22), that is the existence of  $N$ -independent static factors, see the seminal papers Stock and Watson (2002a,b), Bai and Ng (2002), Bai (2003).

As is seen below, Assumption 3 and representation (3.21)–(3.22) are equivalent. That Assumption 3 implies representation (3.21)–(3.22) has been proved above. Conversely, if  $(f_t)$  fulfills equation (3.22), then it has the minimal state-space representation

$$\begin{aligned}\tilde{x}_{t+1} &= \tilde{F}\tilde{x}_t + \tilde{G}v_t \\ f_t &= \tilde{H}\tilde{x}_t,\end{aligned}$$

where  $\tilde{x}_t$  is a  $\tilde{n}$ -dimensional state vector. Thus the  $N$ -dimensional vector  $\chi_t$  has the minimal state-space representation

$$\begin{aligned}\tilde{x}_{t+1} &= \tilde{F}\tilde{x}_t + Gv_t \\ \chi_t &= L\tilde{H}\tilde{x}_t,\end{aligned}$$

where  $\tilde{n}$  is independent of  $N$ , so that representation (3.21)–(3.22) implies Assumption 3.

Another assumption on the static factors is usually imposed, i.e. that the first  $r$  eigenvalues of the covariance matrix  $\gamma_\chi(0)$  diverge as  $N \rightarrow \infty$ . Its introduction and motivation are better discussed in Section 4.1, where we deal with estimation of DFM's.

The advantage of a static factor process  $(f_t)$  for  $(\chi_t)$  is that modeling the dynamics of  $(\chi_t)$  can be done by modeling the dynamics of  $(f_t)$ , so that the dimension of the parameter space can be reduced and is independent of  $N$ . However, even for  $q < N$  we may have  $r = N$ , see Section 4.1. This is the case when there is no non-trivial constant (i.e. independent of  $\lambda$ ) element in the left kernel of  $f_\chi(\lambda)$ , as this is equivalent to

$$\gamma_\chi(0) = \int_{-\pi}^{\pi} f_\chi(\lambda)d\lambda$$

being nonsingular (for more details see Deistler (2019)).

### 3.3 Singular ARMA systems: The genericity of the AR case

In this section we explain that for the case  $r > q$  “generically”, in a sense to be described below, the static factor process  $(f_t)$  in (3.22) is an AR process. This

is important because estimation in the AR case is much simpler compared to the ARMA case.

Consider the ARMA system (3.22)

$$\alpha(z)f_t = \beta(z)v_t,$$

where  $r > q$  and thus  $\beta(z)$  is a “tall” matrix. As is intuitively clear,  $\beta(z)$  is generically zeroless (i.e. in the Smith-McMillan form (3.15) the  $\epsilon_i$  are generically constant), since e.g. the zeros of a suitably chosen nonsingular  $q \times q$  submatrix of  $\beta(z)$  “typically” can be compensated by another  $q \times q$  submatrix of  $\beta(z)$ . In other words, and to be more precise, for given  $p$  and  $m$  (the orders of the AR and MA matrix polynomials respectively), generically, i.e. for an open and dense subset in the parameter space,  $\beta(z)$  has no zeros. This implies that generically the ARMA system (3.22) fulfills the minimum phase condition or that, equivalently,  $v_t$  is an innovation process for  $f_t$ .

As is well known, every zeroless  $r \times q$  polynomial matrix, with  $q < r$ , can be extended to a unimodular  $r \times r$  matrix  $\tilde{\beta}(z) = (\beta(z) \delta(z))$ , say. Now write (3.22) as

$$\tilde{\beta}^{-1}(z)\alpha(z)f_t = \begin{pmatrix} v_t \\ 0 \end{pmatrix}. \quad (3.23)$$

Since  $\tilde{\beta}(z)$  is unimodular,  $\tilde{\beta}^{-1}(z)$  is unimodular too and thus (3.23) is a (singular) autoregression, additionally satisfying the stability condition (3.2). Multiplying both sides of (3.23) by  $(\beta(0) \delta(0))^{-1}$ , which is of course nonsingular, we obtain:

$$a(z)f_t = bv_t, \quad a(z) = I_r - \sum_{j=0}^{\tilde{p}} A_j z^j, \quad b \in \mathbb{R}^{r \times q}, \quad (3.24)$$

where  $b = \beta(0)$ , see e.g. Anderson and Deistler (2008), Anderson et al. (2016). Thus, in a certain sense, for  $r > q$ , “almost every” factor process ( $f_t$ ) can be assumed to be generated by a singular AR process.

As opposed to regular AR processes (where  $r = q$  and  $b = I$  hold), here the assumption  $a(0) = I$  does not guarantee identifiability, unless the assumptions that  $(a(z) b)$  is left coprime and that  $\text{rank}(A_{\tilde{p}}, b) = r$  are imposed.

In this AR setting, the factor process is described by the integer-valued parameters  $r$ ,  $q$  and  $\tilde{p}$  (the latter being the autoregression order) and the parameter space guaranteeing identifiability is

$$\Theta = \{ \text{vec}(A_1, \dots, A_{\tilde{p}}, b), \text{ where (3.2) holds, } A_{\tilde{p}} \text{ and } b \text{ are left coprime and } \text{rank}(A_{\tilde{p}}, b) = r \}. \quad (3.25)$$

However, not every singular AR system can be described in such a parameter space. For more general parameter spaces see Deistler et al. (2011), where it has been shown that by prescribing the column degrees of  $a(z)$  and by assuming  $\text{rank}(A_c, b) = r$ , where  $A_c$  denotes the column end matrix, every singular AR system can be parameterized (see also Section 4.3.2).

It is important to point out that  $r > q$  has been invariably observed in empirical applications of DFM's to large macroeconomic datasets, see Barigozzi et al. (2021) for a review. A very interesting consequence of singularity is briefly accounted for in Section 5.1.

## 4 Separation of the Common Components. Estimation

### 4.1 Principal components

We start by illustrating estimation of the latent variables  $\chi_{it}$ , given the observables  $y_{it}$ , by means of this simplest example. Assume that

$$y_{it} = \chi_{it} + \xi_{it}, \quad \chi_{it} = L_i v_t, \quad (4.1)$$

where  $(v_t)$  and  $(\xi_{it})$  are scalar unit-variance white noise processes, fulfilling (2.2) and (2.3). Moreover, assume that the processes  $(\xi_{it})$  are mutually orthogonal at all leads and lags. We have:

$$\begin{aligned} \gamma_\chi(0) &= \text{E} \chi_t \chi_t' = (L_1 \ L_2 \ \cdots \ L_N)' (L_1 \ L_2 \ \cdots \ L_N) \\ \gamma_y(0) &= \text{E} y_t y_t' = \gamma_\chi(0) + I_N \end{aligned}$$

The model is static and  $f_\chi(\lambda) = (2\pi)^{-1} \gamma_\chi(0)$  for all  $\lambda$ 's. All assumptions 1 through 4 and 6 are obviously fulfilled. Regarding Assumption 5, the rank of  $f_\chi(\lambda)$  is 1 so that  $q = 1$ . We have

$$\omega_{\chi,1}^N = (2\pi)^{-1} \sum_{i=1}^N L_i^2,$$

(which is independent of  $\lambda$ ) so that assuming that  $\sum L_i^2 \rightarrow \infty$ , Assumption 5 is also fulfilled. Lastly, observe that the first eigenvalue of the matrix  $\gamma_y(0)$  is

$$\mu_{y,1}^N = (2\pi) \omega_{\chi,1}^N + 1 = \sum_{i=1}^N L_i^2 + 1,$$

with left eigenvector  $(L_1 \ L_2 \ \cdots \ L_N)$ . Now consider the following average of the observable variables  $y_{it}$ ,  $i = 1, \dots, N$ :

$$p_{y,t}^N = \frac{1}{\sqrt{\mu_{y,1}^N}} \sum_{i=1}^N L_i y_{it},$$

which is known as the first principal component of the  $N$ -dimensional vector with coordinates  $y_{it}$ ,  $i = 1, \dots, N$ , and define

$$A_{y,t}^N = \frac{1}{\sqrt{\mu_{y,1}^N}} p_{y,t}^N. \quad (4.2)$$

We see that

$$\begin{aligned} A_{y,t}^N &= \frac{1}{\mu_{y,1}^N} \sum_{i=1}^N L_i \chi_{it} + \frac{1}{\mu_{y,1}^N} \sum_{i=1}^N L_i \xi_{it} = \frac{1}{\mu_{y,1}^N} \sum_{i=1}^N L_i^2 v_t + \frac{1}{\mu_{y,1}^N} \sum_{i=1}^N L_i \xi_{it} \\ &= \frac{\sum_{i=1}^N L_i^2}{\sum_{i=1}^N L_i^2 + 1} v_t + \frac{1}{\sum_{i=1}^N L_i^2 + 1} \sum_{i=1}^N L_i \xi_{it}, \end{aligned}$$

which implies that

$$E(A_{y,t}^N - v_t)^2 = \left[ \frac{\sum_{i=1}^N L_i^2}{\sum_{i=1}^N L_i^2 + 1} - 1 \right]^2 + \frac{\sum_{i=1}^N L_i^2}{\left[ \sum_{i=1}^N L_i^2 + 1 \right]^2},$$

and therefore that the limit in mean square of  $A_{y,t}^N$ , as  $N \rightarrow \infty$ , is  $v_t$ . Also, in mean square, the projection of  $y_{it}$  on  $A_{y,t}^N$  converges to  $\chi_{it}$  and the regression coefficient to  $L_i$ .

Example (4.1) illustrates the basic features of the estimation techniques used in DFM's. The weighted average  $A_{y,t}^N$ , which is the rescaled first principal component of the observable variables  $y_{it}$ , does both the "cleaning" of the  $y$ 's, in that it averages out the idiosyncratic components, and the *consistent estimation* of the common components.

However, the estimate  $A_{y,t}^N$  is defined using the population covariances of the  $y$ 's—as though, so to speak,  $T$  were infinite—and is therefore unfeasible. In empirical situations, such covariances are estimated and depend both on  $N$  and  $T$ . Moreover, the principal-component technique must be extended to the general case, in which, firstly,  $r$ , the number of static factors, can be greater than  $q$  and  $q$  can be greater than unity and, secondly, the static factors depend on the dynamic factors through an ARMA model. Before we go over the estimation procedure in the general case we must however introduce another assumption.

**Assumption 8.** *The  $r$  largest eigenvalues of the covariance matrix  $\gamma_\chi(0)$  diverge as  $N \rightarrow \infty$ .*

We show below, by means of an example, that Assumption 8 is not a consequence of Assumption 5 and that it is necessary for consistent estimation of the space spanned by the factors  $f_t$ .

Note that if the static factors are orthonormal,  $\gamma_\chi(0) = LL'$ . In any case, as the covariance matrix of  $f_t$  is nonsingular, Assumption 8 is equivalent to assuming that the first  $r$  eigenvalues of  $LL'$  diverge as  $N \rightarrow \infty$ .

Under the assumptions 1 through 8, the first step of the procedure consists in the estimation of the integers  $q$  and  $r$ . This is a non-standard problem because, firstly, both  $T$  and  $N$  tend to infinity, and, secondly, the factors are estimated, not observed. Bai and Ng (2002) provide a class of information criteria allowing to consistently estimate  $r$ . In the same vein see Hallin and Liška (2007) for the estimation of  $q$ .

The second step estimates the static factors  $f_t$ . By (2.1) and (3.21), setting  $L_i = (L_{i1} \ L_{i2} \ \cdots \ L_{ir})$ ,

$$y_{it} = L_i f_t + \xi_{it} = L_{i1} f_{1t} + L_{i2} f_{2t} + \cdots + L_{ir} f_{rt} + \xi_{it}.$$

The first  $r$  principal components of the observable variables  $y_{it}$ , rescaled as in (4.2), are computed, based on their estimated covariances, and used to estimate the space spanned by the factors  $f_t$ . The common components  $\chi_{it}$  and the loadings  $L_i$  are estimated by regressing the  $y$ 's on the estimated factor space. Bai (2003) proves, under some additional technical assumptions, that these estimates of the common components converge in probability to their population counterparts, as both  $N$  and  $T$  tend to infinity, with rate

$$\max \left( \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right).$$

In the third step an ARMA model for the estimated static factors is estimated, see equation (3.22), this leading to the estimation of the dynamic factors as the innovations of the ARMA model. When  $r > q$  the ARMA can be replaced by a singular  $AR$ , see Section 3.3. For this approach see Forni et al. (2009).

Some observations are in order.

(I) Firstly, let us show by an example that Assumption 8, which is the static counterpart of Assumption 5, is necessary for consistent estimation by means of principal components. Let us slightly modify model (4.1) in the following way:

$$\begin{aligned} y_{1t} &= v_t + v_{t-1} + \xi_{1t} \\ y_{it} &= v_t + \xi_{it}, \quad \text{for } i > 1. \end{aligned} \tag{4.3}$$



The space  $\mathcal{S}_t^N$ , spanned by  $\chi_{it}$ ,  $i \leq N$ , for  $N > 1$ , has dimension 2, so that  $r = 2$ , a basis being  $f_{1t} = v_t$ ,  $f_{2t} = v_{t-1}$ . It is fairly easy to see that both the first eigenvalue of  $\gamma_\chi(0)$  and of  $f_\chi(\lambda)$  diverge at rate  $N$ . However, the second eigenvalue of  $\gamma_\chi(0)$  is bounded. Thus Assumption 5 holds with  $q = 1$  but Assumption 8 does not. It is also easy to see that the first rescaled principal component converges to  $v_t$ , but the second does not “clean” the variables  $y_{it}$  from the idiosyncratic component and therefore does not converge to the space spanned by the factors. The consequence is that the common and idiosyncratic components of  $y_{it}$ , as estimated by principal components, are, respectively:

$$\begin{aligned} v_t \text{ and } v_{t-1} + \xi_{1t} \text{ for } i = 1, \\ v_t \text{ and } \xi_{1t} \text{ for } i > 1. \end{aligned}$$

For another example consider

$$\chi_{it} = v_t + M_i v_{t-1} \tag{4.4}$$

with  $\sum_{i=1}^{\infty} M_i^2 < \infty$ . In this case the common and idiosyncratic components of  $y_{it}$  estimated by the principal components are  $v_t$  and  $M_i v_{t-1} + \xi_{it}$ , respectively.

(II) What examples (4.3) and (4.4) show is that in order to “finally” (i.e. for  $N \rightarrow \infty$ ) separate the idiosyncratic from the common components it is necessary that each factor is loaded infinitely often and that the loading coefficients, if declining, do not decline too fast. This is usually rendered by saying that the factors must be “pervasive” and has a precise formulation in Assumption 8.

## 4.2 Generalized Dynamic Factor Models

Let us conclude this section by mentioning a strand of literature in DFMs in which Assumption 3 does not necessarily hold. As we have seen, Assumption 3 is equivalent to assuming that the dynamics of the latent variables  $\chi_{it}$  are completely accounted for by the dynamics of the finite-dimensional,  $N$ -independent vector  $f_t$ , via the static loadings  $L_i$ . The following factor model is an elementary example in which no static factors exist. Let

$$\chi_{it} = \frac{1}{1 - \alpha_i z} v_t, \tag{4.5}$$

where  $v_t$  is scalar unit-variance white noise,  $-0.8 \leq \alpha_i \leq 0.8$ ,  $\alpha_i \neq \alpha_j$  for all  $i$  and  $j$ ,  $i \neq j$ . We have  $f_\chi(\lambda) = (2\pi)^{-1} H_N(\lambda) H_N^*(\lambda)$ , where

$$H_N(\lambda) = ((1 - \alpha_1 e^{-i\lambda})^{-1} \ \dots \ (1 - \alpha_N e^{-i\lambda})^{-1})'$$

As  $f_\chi(\lambda)$  has rank one for all  $\lambda$ , the first eigenvalue of  $f_\chi(\lambda)$  is its trace, that is

$$\omega_{\chi,1}^N(\lambda) = (2\pi)^{-1} \sum_{j=1}^N |1 - \alpha_j e^{-i\lambda}|^{-2}.$$

Because  $\omega_{\chi,1}^N(\lambda)$  diverges for all  $\lambda$ , we have  $q = 1$ . On the other hand,

$$\chi_{it} = v_t + \alpha_i v_{t-1} + \dots + \alpha_i^{N-1} v_{t-N+1} + \dots.$$

If the matrix  $\gamma_\chi(0) = E\chi_t\chi_t'$  were singular then the  $N \times N$  matrix with  $\alpha_i^{m-1}$  in entry  $(i, m)$ , with  $i, m = 1, \dots, N$  should be singular. But the determinant of the latter is the Vandermonde determinant of  $\alpha_1, \dots, \alpha_{N-1}$ , which vanishes only if at least two of the  $\alpha$ 's are equal. Thus the dimension of the space  $\mathcal{S}_t^N$  is  $N$ , not some  $N$ -independent  $r$ , and Assumption 3 does not hold. As a consequence the estimation technique based on a fixed finite number of principal components does not apply.

The DFM without Assumption 3 has been studied by means of frequency-domain methods in Forni et al. (2000), Forni and Lippi (2001), Hallin and Lippi (2013), Forni et al. (2015), Forni et al. (2017), Forni et al. (2018), and called Generalized Dynamic Factor Model. The main tool is the dynamic principal component analysis introduced in Brillinger (1981), which consist of linear combinations of current, past and future values of the observable variables  $y_{it}$  (instead of just current values as in the standard principal components).

We cannot discuss here the merits of this ‘‘dynamic’’ approach relative to the one adopted in the present paper. We limit ourselves to observing that by means of the dynamic principal components the latent variables in model (4.5) can be consistently estimated. Moreover, by means of the dynamic principal components, the common and idiosyncratic components of the variable  $y_{1t}$  in example (4.3) would be correctly estimated as  $v_t + v_{t-1}$  and  $\xi_{1t}$  respectively. The same holds for example (4.4), where by means of the dynamic principal components we estimate the latent variables  $v_t + M_i v_{t-1}$ . Thus the approach based on the dynamic principal components gives the correct results even when Assumption 3 does not hold, or when Assumption 3 holds but not Assumption 8.

### 4.3 A State-Space Formulation of a DFM. Generic Identifiability and Maximum Likelihood Estimation

#### 4.3.1 The State-Space Formulation

A different approach to estimation of DFM's has been introduced in Doz et al. (2012). The paper employs a maximum likelihood estimator for the DFM resulting

from the assumption that the idiosyncratic components are cross-sectionally uncorrelated, and shows that this misspecification has no effect on the estimated common components as  $N \rightarrow \infty$ . See also Bai and Li (2016), Barigozzi and Luciani (2019) and Poncela et al. (2021). This motivates the following formulation of a DFM in state space.

To repeat, it is assumed that the underlying model is an exact factor model, i.e. that the univariate idiosyncratic components are mutually uncorrelated; in addition we assume that they are of AR(1) type (the latter can easily be generalized). Both assumptions of course restrict generality, but are nevertheless appropriate for many applications. An advantage of the state-space formulation is that an EM algorithm of Shumway-Stoffer type, see Shumway and Stoffer (2000), can be used for parameter estimation by means of the Kalman smoother. Clearly in this case identifiability is an important advantage.

We retain the assumption that  $N > r \geq q$  and as earlier, we have the latent variables and minimal static factors related by (3.18). Further, in case  $r > q$ , and relying on an assumption of genericity, there is no loss of generality in working with an AR model for the minimal static factor process as given by (3.24). Indeed, even if  $r = q$ , we shall assume that such a model is valid. This, of course, is not a consequence of genericity, and is restrictive.

Next, we shall assume that the  $i$ -th entry of the idiosyncratic component,  $\xi_{it}$ , is the first order AR process:

$$\xi_{i,t} = \delta_i \xi_{i,t-1} + \eta_{it} \quad (4.6)$$

where  $|\delta_i| < 1$ ,  $i = 1, 2, \dots, N$ , and  $(\eta_{it})$  are mutually uncorrelated zero-mean white noise processes, and also uncorrelated with the process  $v_t$  driving the factor process model.

These assumptions follow the construction of a state-space model, where the  $(r\tilde{p} + N)$ -dimensional state vector is taken to be

$$x_t = \begin{bmatrix} f_t \\ f_{t-1} \\ \vdots \\ f_{t-\tilde{p}+1} \\ \xi_t \end{bmatrix} \quad (4.7)$$

The model is given by

$$\begin{aligned}
x_{t+1} &= \begin{bmatrix} A_1 & A_2 & \dots & A_{\bar{p}-1} & A_{\bar{p}} & 0 \\ I & 0 & \dots & 0 & 0 & 0 \\ 0 & I & \dots & 0 & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \delta \end{bmatrix} x_t + \begin{pmatrix} b & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \\ 0 & 0 \\ 0 & I_N \end{pmatrix} \begin{pmatrix} v_{t+1} \\ \eta_t \end{pmatrix} \\
&= Ax_t + B \begin{pmatrix} v_{t+1} \\ \eta_t \end{pmatrix} \\
y_t &= [L \ 0 \ I_N]x_t = Cx_t
\end{aligned} \tag{4.8}$$

Here,

$$\delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_N), \quad \eta_t = (\eta_{1t} \ \eta_{2t} \ \dots \ \eta_{Nt})', \quad \xi_t = (\xi_{1t} \ \xi_{2t} \ \dots \ \xi_{Nt})'$$

and, to repeat,

$$a(z)f_{t+1} = bv_{t+1}, \quad a(z) = I_r - \sum_{j=1}^{\bar{p}} A_j z^j. \tag{4.9}$$

Of course, we retain the stability requirement that  $\det(I_r - \sum_j A_j z^j) \neq 0$  for  $z \leq 1$ .

Note that the dimension of the state depends on  $N$  and that this may cause problems in proving consistency, see Bańbura and Modugno (2014).

### 4.3.2 Generic Identifiability

In studying identifiability of such a model, one should eliminate unnecessary parameters. Hence we shall assume, as in Section 3.2, the standard normalization for  $L$  (the top  $r \times r$  submatrix of  $L$  is equal to  $I_r$ ), in order to uniquely obtain  $L$  from  $E\chi_t\chi_t'$ .

We shall assume, using a further appeal to genericity, that none of the quantities  $\delta_i^{-1}$  is a zero of  $\det(I_r - \sum_j A_j z^j)$ .

The first step in establishing generic identifiability, is to explain how the *separation of common and idiosyncratic components* can be achieved, or, equivalently, how we can separate the spectrum matrix  $f_y$  into its two additive components  $f_\chi$  and  $f_\xi$ . There are in fact two ways in which this can be done.

First, since the power spectrum  $f_y$  is rational, it has a partial fraction expansion. By genericity, each pole is simple. Each  $\delta_i$  gives rise to a pole  $\delta_i^{-1}$  and appears in the  $i$ -th diagonal entry of  $f_\xi$ , but not in  $f_\chi$ , for which all poles are zeros of  $\det(I_r - \sum_j A_j z^j) \neq 0$ . Hence the residue matrix associated with the pole  $\delta_i^{-1}$  in the

partial fraction expansion of  $f_y$  is a diagonal matrix of rank 1. On the other hand, the residue matrix associated with any pole arising as a zero of  $\det(I_r - \sum_j A_j z^j) \neq 0$  will generically be a matrix with many, and maybe all, nonzero entries, even should it have rank 1. Hence the power spectrum  $f_\chi$  can be determined by adding together those summands of the partial fraction expansion of  $f_y$  whose residue matrices are other than diagonal and of rank 1.

For the alternative procedure, let us suppose that  $N$  exceeds  $2q$ . We can expect by genericity that  $q \times q$  submatrices of  $f_\chi$  obtained by deleting an arbitrary set of  $N - q$  columns and an arbitrary set of  $N - q$  rows are nonsingular, while  $(q + 1) \times (q + 1)$  submatrices obtained via a like process are necessarily singular. Now for each diagonal entry of  $f_y$ , choose a  $(q + 1) \times (q + 1)$  matrix of  $f_y$  by selecting  $(q + 1)$  not necessarily contiguous rows and  $(q + 1)$  not necessarily contiguous columns containing that diagonal entry but containing no other diagonal entry. Note that such a choice is possible precisely because  $N > 2q$ . The entries of the submatrix will be identical with the entries of the corresponding submatrix of  $f_\chi$ , save for the entry corresponding to the single diagonal entry of  $f_y^N$ . Singularity of the submatrix of  $f_\chi$  for which all but one entry are known will allow identification of the remaining entry, which is a diagonal entry of  $f_\chi$ . Since all diagonal entries of  $f_\chi$  can be obtained this way, and the off-diagonal entries of the matrix are identical with those of  $f_y$ , again the separation is achieved.

The next step in establishing generic identifiability is to construct the “real-valued” parameters  $L$ ,  $A_1, \dots, A_{\tilde{p}}$ ,  $b$ ,  $\delta$  and  $E\eta_{it}^2$ , for given integral-valued specification parameters  $r$ ,  $q$ ,  $\tilde{p}$  from the given spectral density  $f_\chi$ , or, equivalently, from the second moments of  $(\chi_t)$ . This is done as follows:

1. From  $E\chi_t\chi_t' = LL'$ ,  $L$  can be uniquely determined using the standard normalization introduced in Section 3.2.
2. The transformation (3.19) then uniquely defines the second moments of the process  $(f_t)$ .
3. Now consider the autoregression (3.24); then, as well known, the parameters  $A_1, \dots, A_{\tilde{p}}$ ,  $b$  are uniquely defined from the (population) second moments of  $(f_t)$  if the following assumption holds:

**Assumption 9.**

$$E \begin{pmatrix} f_{t-1} \\ \vdots \\ f_{t-\tilde{p}} \end{pmatrix} \begin{pmatrix} f_{t-1} \\ \vdots \\ f_{t-\tilde{p}} \end{pmatrix}' > 0 \quad (4.10)$$

*holds.*

4. Finally, from (2.4) we obtain  $f_\xi$  from  $f_y$  and  $f_\chi$  and thus the parameters  $\delta$  and  $E\eta_{it}^2$ .

Note that Assumption 9 is equivalent to controllability of (4.8). Due to our assumptions, (4.10) is fulfilled, as easily shown, for  $r = q$ . For  $r > q$ , however, which in a certain sense is standard, this may not be the case. As shown in Deistler et al. (2011), see p. 20, in this case a first basis of elements of  $(f'_t, \dots, f'_{t-\tilde{p}+1})$  can be selected and this corresponds to a prescription of column degrees  $p_i \leq \tilde{p}$ ,  $i = 1, \dots, r$ , for  $a(z)$  in (4.9). With the corresponding prescription of a state vector, this modified state space system is controllable (for this argument see also the comment on (3.25) in Section 3.3).

## 5 Macroeconomic Applications: Some Consequences of Singularity

A large literature has used DFM's as a tool for forecasting key macroeconomic indicators, see the seminal papers Stock and Watson (2002a,b), see also Forni et al. (2005) and Stock and Watson (2016). In another important application DFM's have been used in structural macroeconomic analysis. It has been shown that by replacing the macroeconomic variables of interest with their common components, estimated from a large dataset by the DFM technique, provides a solution to a much-debated difficulty known among macroeconomists as the "fundamentalness problem". Such solution, as we see below, depends on the singularity of the static factors and the results presented in Section 3.3.

Interesting issues, arising with nonstationarity of the variables  $y_{it}$ , which is of course the case for the majority of the macroeconomic variables, are briefly introduced in Section 5.2.

### 5.1 Applications to Structural Macroeconomic Analysis

We give here a short illustration of this literature by means of a very simple example. Consider a DFM with  $q = 1$  and suppose that

$$y_{it} = v_t + M_i v_{t-1} + \xi_{it}. \quad (5.1)$$

Then focus on the vector  $(\chi_{1t} \ \chi_{2t})'$ :

$$\begin{aligned} \chi_{1t} &= v_t + M_1 v_{t-1} = (1 + M_1 z) v_t \\ \chi_{2t} &= v_t + M_2 v_{t-1} = (1 + M_2 z) v_t. \end{aligned} \quad (5.2)$$

This is a singular vector and we see that the  $2 \times 1$  matrix

$$\begin{pmatrix} 1 + M_1 z \\ 1 + M_2 z \end{pmatrix}$$

is zeroless unless  $M_1 = M_2$ , and thus generically zeroless as  $(M_1 \ M_2)$  varies in an open set of  $\mathbb{R}^2$ . It is convenient to exclude from the parameter space all points  $(M_1 \ M_2)$  with  $|M_1| = 1$  or  $|M_2| = 1$ .

Thus generically the minimum phase condition is fulfilled for (5.2), or, in an alternate terminology,  $v_t$  is *fundamental* in (5.2). Note that this does not imply that  $|M_1| < 1$  or  $|M_2| < 1$ . In other words, if  $M_1 \neq M_2$ ,  $v_t$  is fundamental for the 2-dimensional vector  $(\chi_{1t})$  even though it is non-fundamental for each of the scalar processes  $\chi_{1t}$  and  $\chi_{2t}$  taken separately.

Now suppose that an econometrician is interested in  $y_{1t}$  and, for the sake of simplicity, that  $y_{1t}$  is observed without error, i.e.  $y_{1t} = \chi_{1t}$ . We assume also that  $y_{1t} = v_t + M_1 v_{t-1}$  is a structural equation, i.e. that the parameter  $M_1$  and the white noise  $v_t$  have a structural interpretation.

Standard Var analysis would estimate a VAR for  $y_{1t}$ , which is just an AR in this case, then the AR would be inverted. As the generating process is an MA(1), this procedure estimates consistently an MA(1):

$$y_{1t} = w_t + N_1 w_{t-1}.$$

Now,  $w_t$ , being the residual of an AR, is an innovation for  $y_{1t}$ , which implies that  $|N_1| < 1$ . Thus  $N_1$  is equal to  $M_1$  only if  $|M_1| < 1$ , otherwise  $N_1 = 1/M_1$ . The so-called *fundamentalness problem* in Structural VAR analysis arises because usually the econometrician's information is not sufficient to identify the structural model among those consistent with the spectral density of the observable vector. In our case the econometrician is not able to decide between  $N_1$ , which is by definition less than unity in modulus, and  $1/N_1$ .

The solution of the fundamentalness problem based on DFM's can be presented, in the case of our simple example, as follows:

1. We have assumed that  $y_{1t}$ , the variable of interest, belongs to a large macroeconomic dataset  $(y_{it})$ ,  $i = 1, \dots, N$ .
2. Assuming that the variables in the dataset have the DFM structure 5.1, with  $q = 1$  and  $r = 2$ , we apply the separation-estimation technique outlined in Section 4, thus obtaining the static factors  $f_t$ , the loadings  $L_i$  and the common components  $\chi_{it}$ .
3. Now consider any 2-dimensional vector  $(\chi_{1t} \ \chi_{it}) = (y_{1t} \ \chi_{it})$ , with  $i \neq 1$ , for example  $\chi_t = (y_{1t} \ \chi_{2t})$ . An estimate of a singular VAR for  $\chi_t$  and its inversion will consistently estimate a vector MA(1) for  $\chi_t$ , with a white noise  $w_t$  which is fundamental. On the other hand,  $v_t$  is generically fundamental in (5.2). Uniqueness of fundamental representations implies that  $w_t = v_t$  and the first equation in the estimated vector MA(1) is precisely  $y_{1t} = v_t + M_1 v_{t-1}$ .

Note that in step 3 we estimate a VAR for the common components of  $\chi_{1t}$  and  $\chi_{2t}$ . Alternatively, we can estimate a singular VAR for the factors. For details in the general case and macroeconomic applications see Forni et al. (2009), Stock and Watson (2016), Forni et al. (2020).

## 5.2 Nonstationary DFMs and cointegration of the factors

In general only some of the processes in an empirical dataset are stationary. Assuming that the nonstationary processes are I(1), the separation-estimation procedure described in Section 4 applies to the dataset obtained by taking first differences of the I(1) processes.

Suppose for simplicity that all the processes  $y_{it}$ ,  $\chi_{it}$ ,  $f_t$  and  $\xi_{it}$  are I(1). We consistently estimate  $(1-z)f_t$ ,  $(1-z)\chi_{it}$  and  $(1-z)\xi_{it}$ . Then the levels are obtained by integration. This, apart from minor issues regarding the initial conditions, is a fairly trivial extension. However, if we want to estimate a VAR for the factors  $f_t$  or a vector of common components, as in Section 5.1, cointegration must be taken into account. Indeed, under the assumption  $r > q$ , i.e. under singularity of  $f_t$ , the spectral density of  $f_t$  has rank  $q$  at all frequencies and therefore at frequency zero, so that  $f_t$  is cointegrated with cointegration rank at least  $r - q$ .

For cointegration of singular vector processes and the singular version of the Granger representation theorem, see Deistler and Wagner (2017), Barigozzi et al. (2020). See also Barigozzi et al. (2021) for estimation and some empirical applications.

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