TIME SERIES

Midterm assignments (2020 fall semester)

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You are assumed to choose between the following midterm assignments (each of them is for 50 points maximum, and you must have minimum 20 for the signature):

- Solve at least 4 of the theoretical Exercises below. Computer aided results are not accepted, you have to turn in detailed calculations in handwriting or in LaTex. It is important that you send a pdf fie and I could see the detailed calculations in it.

- Find a paper (see a possible bibliographic list below) and prepare a 2-3 pages long description about the most important issues, theorems in it, possibly with proof. Also give a 10 minutes presentation on it. You can choose a shorter paper or part of a longer one. You can work in pairs as well, but the collaborating students are supposed to work on disjoint parts of a paper. Other papers are also welcome with some math content.

- Perform time series analysis on a real-life dataset, and explain your results in a 10 minutes presentation. Also prepare a 2-3 pages description. You can find data and programs in R or Python, but you can use any other data you have or find in the internet.

Please, write me (in e-mail) until the end of October, which option you choose, and in case of presentation, please give me the paper title you select. Then you are supposed to send me the solutions (if you solve exercises) or the writeup of presentation or data processing before 15 November, so that I can make a schedule of talks. The last week of the semester is for a written exam, the result of which can be improved by an oral one in the exam period.

Theoretical exercises

1. Let $\{\xi_j\}_{j\in\mathbf{Z}}$ be a sequence of i.i.d. $\mathcal{N}(0,\sigma^2)$ (Gaussian) random variables and let a, b, c be real constants. Which of the following processes are weakly stationary? Why? Find their mean and autocovariance function too:

   (a) $X_t = a + b\xi_t + c\xi_{t-2}$

   (b) $X_t = \xi_0 \cos(ct)$

2. Let $\{\xi_j\}_{j\in\mathbf{Z}}$ be a sequence of i.i.d. $\mathcal{N}(0,\sigma^2)$ (Gaussian) random variables and let a, b, c be real constants. Which of the following processes are weakly stationary? Why? Find their mean and autocovariance function too:

   (a) $X_t = \xi_1 \cos(ct) + \xi_2 \sin(ct)$

   (b) $X_t = \xi_t \cos(ct) + \xi_{t-1} \sin(ct)$

3. Consider the symmetric, 1D random walk defined by

$$X_0 = 0, \quad X_t = \sum_{j=1}^{t} \xi_j, \quad t = 1, 2, \ldots,$$

   where $\{\xi_j\}$ is a binary process: ξ_js are i.i.d. and $\xi_j = \pm 1$ with probability $\frac{1}{2} - \frac{1}{2}$.

   Prove that the random walk process $\{X_t\}$ is not weakly stationary.

4. Consider the 1D, AR(1) process

$$X_t = aX_{t-1} + \xi_t, \quad t = 0, \pm 1, \pm 2, \ldots,$$

   where $\{\xi_t\}$ is a $\mathrm{WN}(0,\sigma^2)$ white noise process and $|a| < 1$ is complex constant. Compute the variance of the sample mean $\frac{1}{4}(X_1 + X_2 + X_3 + X_4)$ if $\sigma^2 = 1$ and

   (a) $a = 0.9$

   (b) $a = -0.9$

5. Consider the 1D, MA(2) process

$$X_t = \xi_t + b\xi_{t-2}, \quad t = 0, \pm 1, \pm 2, \ldots,$$

   where $\{\xi_t\}$ is a $\mathrm{WN}(0,\sigma^2)$ white noise process and b is complex constant. Compute the variance of the sample mean $\frac{1}{4}(X_1 + X_2 + X_3 + X_4)$ if $\sigma^2 = 1$ and

(a) $b = 0.8$

(b) $b = -0.8$

6. Show that the 1D process

$$X_t = A\cos(\omega t) + B\sin(\omega t), \quad t = 0, \pm 1, \pm 2, \ldots$$

is weakly stationary, where A, B are uncorrelated $\mathcal{N}(0,1)$ (standard Gaussian) random variables and ω is a fixed frequency. Find the autocovariance function of the process too.

7. Consider the 1D process

$$X_t = A\cos(\omega t) + B\sin(\omega t), \quad t = 0, \pm 1, \pm 2, \ldots$$

where A, B are uncorrelated $\mathcal{N}(0,1)$ (standard Gaussian) random variables and ω is a fixed frequency. Prove that X_t can be predicted with 0 error, based on its past values.

Hint: try to prove that X_{t-1} and X_{t-2} suffice for the 0 mean square error of the prediction.

8. Show that the following 1D, MA(1) processes have the same autocovariance function:

(a)

$$X_t = \xi_t + b\xi_{t-1}, \quad \xi_t \sim \mathrm{WN}(0, \sigma^2)$$

(b)

$$Y_t = \eta_t + \frac{1}{b}\eta_{t-1}, \quad \eta_t \sim \mathrm{WN}(0, b^2\sigma^2)$$

where $0 < |b| < 1$.

9. Consider the 1D, AR(4) process

$$X_t = 1.4X_{t-1} - 1.1X_{t-2} + 0.4X_{t-3} - 0.1X_{t-4} + \xi_t, \quad t = 0, \pm 1, \pm 2, \ldots,$$

where $\{\xi_t\}_{t\in\mathbf{Z}}$ is a $\mathrm{WN}(0, \sigma^2)$ white noise process. Find and sketch the spectral density function $f(\omega)$ of the process together with its local maxima and the corresponding cycle lengths.

10. Consider the 1D, MA(3) process

$$X_t = \xi_{t-1} + \xi_{t-2} + \xi_{t-3}, \quad t = 0, \pm 1, \pm 2, \ldots,$$

where $\{\xi_t\}_{t\in\mathbf{Z}}$ is a $\mathrm{WN}(0, 1)$ white noise process. Find and sketch the autocovariance and the spectral density function $f(\omega)$ of the process.

References

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