Basics: Complex Functions

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1 Holomorphic (or analytic) functions

Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be a complex function. If $z_0 \in \Omega$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists then we denote this limit by $f'(z_0) \in \mathbb{C}$ and call it the *derivative* of f at z_0 and f is said to be *complex differentiable* at z_0. If f is complex differentiable at every $z_0 \in \Omega$, then we say that f is *holomorphic* (or *analytic*) in Ω.

Let $[\alpha, \beta] \subset \mathbb{R}$ be a closed interval. A *path* γ is a piecewise continuously differentiable function $\gamma : [\alpha, \beta] \to \mathbb{C}$. A *closed path* is a path such that $\gamma(\alpha) = \gamma(\beta)$. If $f : \Omega \to \mathbb{C}$ is a continuous complex function and the *range* γ^* of the path γ is in Ω, then the integral of f over γ is defined as

$$\int_\gamma f(z)dz := \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

Let γ be a closed path and take $\Omega = \mathbb{C} \setminus \gamma^*$. Define

$$\mathrm{Ind}_\gamma(z) := \frac{1}{2\pi i} \int_\alpha^\beta \frac{d\zeta}{\zeta - z}, \qquad z \in \Omega. \tag{1}$$

Then Ind_γ is an integer-valued function in Ω which is constant in each component of Ω and which is bounded in the unbounded component of Ω. We call $\mathrm{Ind}_\gamma(z)$ the *index* or *winding number* of z with respect to γ.

The most important special class of closed paths is when there are exactly two components of Ω w.r.t. γ: one where the winding number is 1 and one where the winding number is 0. Such a closed path is called a *simple closed path.* If γ is a simple closed path in Ω, that is, such that

$$\Omega = \Omega_1 \cup \Omega_0 \cup \gamma^*, \quad \mathrm{Ind}_\gamma = 1 \quad \text{in} \quad \Omega_1, \quad \mathrm{Ind}_\gamma = 0 \quad \text{in} \quad \Omega_0, \tag{2}$$

then we may call Ω_1 the *interior* and Ω_0 the *exterior* of γ. This is the case e.g. when $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, so γ^* is the unit circle T. Then γ winds around the points of the open unit disc D exactly once counterclockwise. It is easy to see that in this example $\mathrm{Ind}_\gamma(z) = 1$ if $z \in D$ and $\mathrm{Ind}_\gamma(z) = 0$ if $z \in \{\zeta \in \mathbb{C} : |\zeta| > 1\}$.

The fundamental theorems of complex analysis are *Cauchy's theorem* and *Cauchy's formula.*

Theorem 1. *Suppose that Ω is an open set and f is a holomorphic function in Ω. Then for any closed path γ in Ω such that* $\mathrm{Ind}_\gamma(z) = 0$ *for any $z \notin \Omega$, we have Cauchy's theorem*

$$\int_\gamma f(z)dz = 0.$$

Moreover, we also have Cauchy's formula

$$f(z) \cdot \mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad \forall z \in \Omega \setminus \gamma^*. \tag{3}$$

An important consequence of (3) is that if γ is a simple closed path, then the values of a holomorphic function f in the interior of γ is uniquely determined by the values of f on the boundary γ^*. The condition "$\mathrm{Ind}_\gamma(z) = 0$ for any $z \notin \Omega$" of the theorem prevents a situation that points where f is not holomorphic may influence the value of the integrals. (Imagine that Ω is an annulus and a closed path γ in Ω goes around the inner circle boundary of the annulus.)

Another important consequence is stated in the next theorem.

Theorem 2. *Suppose that Ω is an open set. Then a function f is holomorphic in Ω if and only if it is* representable by power series in Ω *in the sense that for any $a \in \Omega$ there exists an $r > 0$ such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad \forall z \in D(a, r) := \{\zeta : |\zeta - a| < r\}, \tag{4}$$

where each $c_n \in \mathbb{C}$.

It follows that if f is holomorphic in Ω, then f is arbitrary many times complex differentiable in Ω.

Recall that a function f which is representable by power series in an open set Ω is usually called an *analytic function*. We have just seen that the class of holomorphic and the class of analytic functions in an open set Ω coincide in complex analysis. Notice that the power series in (4) contains only nonnegative powers of $(z-a)$. The domain of convergence of such a power series is always a disc $\{z : |z-a| < R\}$ and possibly some points of the boundary circle as well, where $R \in [0, \infty]$,

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}. \tag{5}$$

Here comes another important property of holomorphic functions.

Theorem 3. *(The maximum modulus theorem)*

(a) Suppose that Ω is a connected open set, $D(a,r) \subset \Omega$, $r > 0$, and f is a holomorphic function in Ω. Then

$$|f(a)| \le \max_t |f(a + re^{it})|.$$

Equality occurs here if and only if f is constant in Ω.

(b) Let Ω be a bounded connected open set and let K denote its closure. If the function f is continuous in K and holomorphic in Ω, then

$$|f(z)| \le \sup_{\zeta\in\partial\Omega} |f(\zeta)|, \quad z \in \Omega,$$

where $\partial\Omega$ denotes the boundary of Ω. If equality holds at one point $z \in \Omega$, then f is constant.

Let f be a holomorphic function in the connected open set Ω, and define the *zero set* of f by

$$Z(f) := \{a \in \Omega : f(a) = 0\}.$$

If f is not identically 0 in Ω, then $Z(f)$ has no limit point in Ω and $Z(f)$ is at most countable. Moreover, then to each $a \in Z(f)$ there corresponds a unique positive integer m such that

$$f(z) = (z-a)^m g(z), \qquad z \in \mathbb{C},$$

where g is holomorphic in Ω and $g(a) \neq 0$. Then f is said to have a *zero of order* m at the point a.

Let Ω be an open set, $a \in \Omega$, and f be holomorphic in $\Omega \setminus \{a\}$. Then f is said to have an *isolated singularity* at a. If f can be defined at the point a so that the extended function is holomorphic in Ω, then the singularity is called *removable*. If there is a positive integer m and constants $c_{-1}, \ldots, c_{-m} \in \mathbb{C}$, $c_{-m} \neq 0$, such that

$$f(z) - Q(z) = f(z) - \sum_{k=1}^{m} c_{-k}(z-a)^{-k}$$

has a removable singularity at a, then f is said to have a *pole of order* m at a and $Q(z)$ is called the *principal part* of f at a. Then there exists an $r > 0$ such that f can be expressed by a two-sided power series around a as

$$f(z) = \sum_{k=-m}^{\infty} c_k(z-a)^k, \quad 0 < |z-a| < r.$$

Any other isolated singularity is called an *essential singularity*.

A function f is said to be *meromorphic* in an open set Ω if there is a subset $A \subset \Omega$ such that

1. A has no limit point in Ω,
2. f is holomorphic in $\Omega \setminus A$,
3. f has a pole at each point of A.

The set A can be at most countable. For each $a \in A$, the principal part of f at a has the form

$$Q_a(z) = \sum_{k=1}^{m(a)} c_{-k}^{(a)}(z-a)^{-k};$$

the coefficient $c_{-1}^{(a)}$ is called the residue of f at a:

$$\operatorname{Res}(f, a) := c_{-1}^{(a)}.$$

If γ is a closed path in $\Omega \setminus A$, then elementary integration shows that

$$\frac{1}{2\pi i} \int_{\gamma} Q_a(z)dz = \operatorname{Res}(f; a) \operatorname{Ind}_{\gamma}(a).$$

This simple fact can be used to show the following *Residue theorem*.

Theorem 4. *Suppose that f is a meromorphic function in the open set Ω. Let A be the subset of points at which f has poles. If γ is a closed path in $\Omega \setminus A$ such that* $\mathrm{Ind}_\gamma(z) = 0$ *for all $z \notin \Omega$, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{a\in A} \mathrm{Res}(f;a)\,\mathrm{Ind}_\gamma(a).$$

The next theorem is an application of the Residue theorem; useful to determine how many zeros a holomorphic function f has in the interior of a simple closed path.

Theorem 5. *Assume that γ is a simple closed path in a connected open set Ω, such that* $\mathrm{Ind}_\gamma(z) = 0$ *for any $z \notin \Omega$.*

Let f be a holomorphic function in Ω and let $N_f(\gamma)$ denote the number of zeros of f in the interior Ω_1 of γ, counted according to their multiplicities. Assume that f has no zeros on γ^*. Then

$$N_f(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \mathrm{Ind}_{f\circ\gamma}(0).$$

This theorem is sometimes called the *"Argument principle"*. This name can be explained by a heuristic argument. Since we assumed that f has no zeros on γ^*, along the closed path γ one can take

$$\log f(z) = \log\left(|f(z)|e^{i\arg f(z)}\right) = \log|f(z)| + i\arg f(z),$$

where $\arg f(z)$ denotes the multiple-valued argument (angle) of the complex number $f(z)$. By the chain rule, $(\log f(z))' = f'(z)/f(z)$, and so

$$\begin{aligned}&\int_\gamma \frac{f'(z)}{f(z)}dz = \int_\gamma d\log|f(z)| + i\int_\gamma d\arg f(z)\\ &= \log|f(\gamma(\beta))| - \log|f(\gamma(\alpha))| + i\Delta_\gamma \arg f(z) = i\Delta_\gamma \arg f(z),\end{aligned}$$

since $\gamma(\beta) = \gamma(\alpha)$. Here $\Delta_\gamma \arg f(z)$ denotes the change of argument of f along the closed path γ, which divided by 2π gives the winding number $\mathrm{Ind}_{f\circ\gamma}(0)$ in the theorem.

2 Harmonic functions

If $f : \mathbb{C} \to \mathbb{C}$ is a complex function, we may write that $f(z) = u(x, y) + iv(x, y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are real two-variable functions. If Ω is a plane open set, then f is holomorphic in Ω if and only if u and v are differentiable two-variable functions in Ω and the *Cauchy–Riemann equations* hold:

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v, \quad (x, y) \in \Omega, \tag{6}$$

where ∂_x and ∂_y denote partial differentiation w.r.t. x and y, respectively. Then

$$f' = \frac{1}{2}(\partial_x u + \partial_y v) + \frac{i}{2}(\partial_x v - \partial_y u). \tag{7}$$

Another way of writing the above equalities can be obtained by introducing the differential operators

$$\partial := \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y). \tag{8}$$

Then (6) and (7) are equivalent to

$$\bar{\partial} f = 0, \qquad f' = \partial f. \tag{9}$$

(Applying ∂ and $\bar{\partial}$ goes like multiplication with complex numbers.)

Let Ω be a plane open set and let $u : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable real function such that $\partial_{xx} u$ and $\partial_{yy} u$ exist at every point of Ω. Then the *Laplacian* of u is defined as

$$\Delta u := \partial_{xx} u + \partial_{yy} u.$$

The function u is called *harmonic* in Ω if it is continuous in Ω and $\Delta u = 0$ in Ω.

Similarly, the complex function $f : \mathbb{C} \to \mathbb{C}$, $f = u + iv$, is *harmonic* in Ω if it is continuous in Ω, $\partial_{xx} f$ and $\partial_{yy} f$ exist in Ω, and

$$\Delta f := \partial_{xx} f + \partial_{yy} f = \Delta u + i\Delta v = 0 \quad \text{in} \quad \Omega. \tag{10}$$

If f is a holomorphic function in Ω, then f has continuous derivatives of all orders, so $\partial_{xy} f = \partial_{yx} f$, moreover, $\Delta f = 4\partial\bar{\partial} f$. Since then $\bar{\partial} f = 0$ by (9), it follows that $\Delta f = 0$ in Ω. This shows that holomorphic functions are harmonic. Equation (10) shows that the real and imaginary parts of f are also harmonic, and by the Cauchy–Riemann equations (6) they are strongly related to each other; that is why u and v are called *harmonic conjugates*. For example, any harmonic function u in D is the real part of one and only one holomorphic function $f = u + iv$ such that $f(0) = u(0)$, and then $v(0) = 0$ and this v is also unique.

For $0 \le \rho < 1$; $t, \theta \in \mathbb{R}$ and $z = \rho e^{i\theta}$, the *Poisson kernel* is

$$P_\rho(\theta - t) := \operatorname{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right] = \frac{1 - |z|^2}{|e^{it} - z|^2} = \frac{1 - \rho^2}{1 - 2\rho\cos(\theta - t) + \rho^2}. \tag{11}$$

Then

$$P_\rho(t) > 0, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} P_\rho(t)dt = 1 \quad (0 \le \rho < 1).$$

If $g \in L^1([-\pi, \pi])$, then

$$\tilde{G}(\rho e^{i\theta}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} g(t)dt$$

is a holomorphic function of $z = \rho e^{it}$ in the open unit disc D. Hence the *Poisson integral*

$$G(\rho e^{i\theta}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} P_\rho(\theta - t)g(t)dt \tag{12}$$

is the real part of a holomorphic function, so a harmonic real function in D, for any $g \in L^1([-\pi, \pi])$ real function. It implies that if $g \in L^1([-\pi, \pi])$ is complex valued, the Poisson integral $G(z)$ defined by (12) is a complex harmonic function. Moreover,

$$\lim_{\rho \to 1} G(\rho e^{i\theta}) = g(\theta) \quad \text{in} \quad L^1([-\pi, \pi]). \tag{13}$$

The next theorem gives the unique solution of the *Dirichlet problem*: Assume that a continuous function g is given on T and it is required to find a harmonic function G in D, which is continuous on the closed unit disc $\bar{D}$ and has the boundary values g.

Theorem 6. *(a) Assume that $g \in C(T)$. Let $G(e^{i\theta}) := g(e^{i\theta})$ on T and define $G(z)$ in D by the Poisson integral* (12). *Then G is harmonic in D and $G \in C(\bar{D})$.*

(b) Conversely, suppose that $G \in C(\bar{D})$ and G is harmonic in D. Then G is the Poisson integral (12) *in D of its restriction to T.*

So far the Poisson integral was considered only in the unit disc. However, it is easy to extend it to an arbitrary disc. If g is a continuous complex or real function on the boundary of the open disc $D(a,R) := \{z : |z-a| < R\}$ and if g is defined by the Poisson integral

$$g(a+\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-\rho^2}{R^2-2R\rho\cos(\theta-t)+\rho^2} g(a+Re^{it})dt \tag{14}$$

in $D(a,R)$, then g is continuous on the closed disc $\bar{D}(a,R)$ and harmonic in $D(a,R)$.

Conversely, if u is a harmonic real function in an open set Ω and if $\bar{D}(a,R) \subset \Omega$, then u satisfies (14) in $D(a,R)$ and there is a unique holomorphic function $f = u+iv$ in $D(a,R)$ such that $f(a) = u(a)$ and $v(a) = 0$. In sum, every real harmonic function is locally the real part of a holomorphic function. Consequently, every harmonic function has continuous partial derivatives of arbitrary order.

We say that a continuous complex or real function g has *the mean value property* in an open set Ω if we have

$$g(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(a+Re^{it})dt, \quad \forall \bar{D}(a,R) \subset \Omega. \tag{15}$$

The Poisson integral (14) shows that any harmonic complex or real function has the mean value property. In fact, much more is true, as shown by the next theorem.

Theorem 7. *A continuous complex or real function has the mean value property in an open set Ω if and only if it is harmonic in Ω.*

A real-valued function u defined in a plane open set Ω is said to be *subharmonic* in Ω if it has the following four properties:

1. $-\infty \le u(z) < \infty$ for all $z \in \Omega$;

2. u is upper semicontinuous in Ω;

3. $$u(a) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+Re^{it})dt, \quad \forall \bar{D}(a,R) \subset \Omega; \tag{16}$$

4. none of the integrals above is $-\infty$.

Clearly, every harmonic real function is also subharmonic.

Theorem 8. *(a) If u is subharmonic in the open set Ω and ϕ is a monotonically increasing convex function in $\mathbb{R}$, then $\phi \circ u$ is subharmonic in Ω.*

(b) If Ω is a connected open set in the plane and f is a holomorphic function in Ω which is not identically 0, *then* $\log|f|$, $\log^+|f| := \max(0, \log|f|)$, *and* $|f|^p$ $(0 < p < \infty)$ *are subharmonic in Ω.*

The next theorem explains the term "subharmonic".

Theorem 9. *Suppose that u is a continuous subharmonic function in a plane open set Ω, K is a compact subset of Ω, h is a continuous real function on K which is harmonic in the interior of K, and $u(z) \leq h(z)$ at all boundary points of K. Then $u(z) \leq h(z)$ for all $z \in K$.*

3 Hardy spaces

Let D denote the open unit disc in $\mathbb{C}$ and T be its boundary circle. If $1 \leq s \leq p \leq \infty$, then $L^1(T) \supset L^s(T) \supset L^p(T) \supset L^\infty(T)$. Each $f \in L^1(T)$ has well-defined Fourier coefficients given by

$$a_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt \quad (n \in \mathbb{Z}).$$

For $1 \leq p \leq \infty$ we define the *Hardy space* $H^p(T)$ to be the closed subspace of $L^p(T)$ consisting of all functions for which $a_n = 0$ when $n < 0$. (The map $f \to a_n$ is continuous, this explains why the subspace is closed.) From now on, we disregard the argument T.

Note that H^2 is a Hilbert space. We also define *co-Hardy spaces*

$$\bar{H}^2 := \{f \in L^2(T) : a_n = 0 \text{ if } n > 0\}, \quad \bar{H}_0^2 := \{f \in L^2(T) : a_n = 0 \text{ if } n \geq 0\}.$$

It follows that there is an orthogonal decomposition $L^2(T) = H^2 \oplus \bar{H}_0^2$. Also, $f \in H^2$ if and only if $\bar{f} \in \bar{H}^2$, where $\bar{f}$ is defined by $\bar{f}(e^{it}) := \overline{f(e^{it})}$.

Let us denote the space of square summable complex sequences on $\mathbb{Z}$ by $\ell^2(-\infty, \infty)$. One can define the Fourier transform $\mathcal{F}$ from $\ell^2(-\infty, \infty)$ onto $L^2(T)$ by

$$\mathcal{F}(\{a_n\}_{n \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} a_n e^{int}.$$

Then $\mathcal{F}$ is a unitary map, that is, an isometric isomorphism from $\ell^2(-\infty, \infty)$ onto $L^2(T)$. Let S be the (bilateral) right shift in $\ell^2(-\infty, \infty)$. Then it defines a shift operator in $L^2(T)$ as well by the formula $(Sf)(e^{it}) = e^{it} f(e^{it})$, with which we have

$$\mathcal{F}S = S\mathcal{F}.$$

It is clear that $\mathcal{F}(\ell^2(0, \infty)) = H^2$ and the restricted operators $S|\ell^2(0, \infty)$ and $S|H^2$ are also unitarily equivalent.

There is another useful characterization of outer functions.

Theorem 10. *A function $f \in H^2$ is outer if and only if the linear combinations of the functions $S^n f$ $(n \geq 0)$ are dense in H^2.*