Lessons 1-2: Multivariate Stationary Time Series (Autocovariance Matrices and Spectral Representation)

Marianna Bolla and Tamás Szabados, BME Math. Inst.

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1 Covariance matrix function and spectral measure

Definition 1. *The d-dimensional time series $\mathbf{X}_t$ ($t \in \mathbb{R}$) is strongly stationary (or stationary in the strong sense) if for any $h \in \mathbb{R}$, $n \in \mathbb{N}$, and time instances $t_1 < t_2 < \dots < t_n$, the joint distribution of $(\mathbf{X}_{t_1+h}, \dots, \mathbf{X}_{t_n+h})$ is the same as the joint distribution of $(\mathbf{X}_{t_1}, \dots, \mathbf{X}_{t_n})$. That is, the joint distributions are invariant for any time shift.*

The next conditions of weak stationarity are simpler to check in practice.

Definition 2. *The d-dimensional time series $\mathbf{X}_t$ ($t \in \mathbb{R}$) (with complex coordinates) is weakly stationary (or stationary in the wide sense) if it has finite expectation and finite covariance function that do not depend on time shift:*

$$\mathbb{E}\mathbf{X}_t = \boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathbb{C}^d,$$
$$c_{jk}(h) := \operatorname{Cov}(X^j_{t+h}, X^k_t) = \operatorname{Cov}(X^j_h, X^k_0) = \mathbb{E}\left((X^j_h - \mu_j)\overline{(X^k_0 - \mu_k)}\right),$$

where $t, h \in \mathbb{R}$; $j, k = 1, \dots, d$, and $\mathbf{X}_t = [X^1_t, \dots, X^d_t]^T$ in terms of its components.

Note that strong stationarity implies the weak one if the process has finite second moments. If the time series is Gaussian then the two notions are equivalent, because Gaussian distributions are uniquely determined by their expectations and covariances. Further, to any weakly stationary process there exists a strongly stationary Gaussian with the same expectation and covariance matrix function. Weakly stationary processes are sometimes called *second order processes*, as only their first and second moments are used in the theory describing their behavior.

Without loss of generality, from now on we assume that $\boldsymbol{\mu} = \mathbf{0}$, in which case

$$\mathrm{Cov}(X^j_{t+h}, X^k_t) := \mathbb{E}(X^j_{t+h}\overline{X^k_t}), \quad \mathrm{Var}(X^k_t) := \mathbb{E}(|X^k_t|^2),$$

where the complex conjugation on the second factor can be disregarded if the components are real valued.

The main object of the present book is time series *with discrete time.*

Definition 3. *The d-dimensional time series $\mathbf{X}_t$ $(t \in \mathbb{Z})$ is weakly stationary (or stationary in the wide sense) with discrete time (and state space $\mathbb{C}^d$) if it has finite expectation and finite cross-covariances that do not depend on time shift:*

$$\mathbb{E}\mathbf{X}_t = \mathbf{0}, \quad c_{jk}(h) := \mathrm{Cov}(X^j_{t+h}, X^k_t) = \mathrm{Cov}(X^j_h, X^k_0) = \mathbb{E}(X^j_h\overline{X^k_0})$$

$(t, h \in \mathbb{Z};\ j, k = 1, \dots, d)$, *where, as we said above, we may assume that* $\mathbb{E}\mathbf{X}_t = \boldsymbol{\mu} = \mathbf{0}$.

From now on, the expression "stationary time series" will refer to a discrete time weakly stationary process with zero expectation, unless it is explicitly stated otherwise.

Considering complex valued random vectors simplifies the discussion; it is easy to describe the specific case of real valued random vectors whenever it is needed. Since we assume finite second moments, it follows that each component X^j_t, $j = 1, \dots, d$, is square integrable, so belongs to the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for any time instant $t \in \mathbb{Z}$.

The relationship between two second order random vectors (of not necessarily the same dimension) is described by their *cross-covariance matrix*:

$$\mathrm{Cov}(\mathbf{X}, \mathbf{Y}) = \mathbb{E}(\mathbf{X}\mathbf{Y}^*) = [\mathbb{E}(X^j\overline{Y^k})]^{d,d'}_{j,k=1} \in \mathbb{C}^{d \times d'}, \tag{1}$$

where $\mathbf{X}$ is a d-dimensional and $\mathbf{Y}$ is a d'-dimensional complex random vector with $L^2(\Omega, \mathcal{F}, \mathbb{P})$ components with zero expectations. Clearly, $\mathrm{Cov}(\mathbf{Y}, \mathbf{X}) =$

$[\mathrm{Cov}(\mathbf{X},\mathbf{Y})]^*$, and $\mathrm{Cov}(\mathbf{X},\mathbf{X})$ is a self-adjoint, non-negative definite (positive semidefinite) $d \times d$ matrix, the usual covariance matrix of $\mathbf{X}$.

We say that $\mathbf{X}$ and $\mathbf{Y}$ are *orthogonal*, denoted $\mathbf{X} \perp \mathbf{Y}$, if $\mathrm{Cov}(\mathbf{X},\mathbf{Y}) = \boldsymbol{O}$, the zero matrix. This is a more general notion of orthogonality, and applicable to random vectors of different dimensions too. Observe that it is a stronger condition than the standard orthogonality $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$ if the two vectors are of the same dimension.

We use the *covariance matrix function* (or *autocovariance matrix function*) $\boldsymbol{C}(h) = [c_{j\ell}(h)]_{d\times d}$ to describe weakly stationary time series,

$$\boldsymbol{C}(h) := \mathrm{Cov}(\mathbf{X}_{t+h}, \mathbf{X}_t) = \mathbb{E}\left(\mathbf{X}_{t+h}\mathbf{X}_t^*\right), \qquad h \in \mathbb{Z}.$$

The covariance matrix function $\boldsymbol{C}$ does not depend on the time instant $t \in \mathbb{Z}$ because of the assumed weak stationarity. Clearly,

$$c_{j\ell}(-h) = \mathbb{E}\left(X_{t-h}^j \overline{X_t^\ell}\right) = \mathbb{E}\left(X_t^j \overline{X_{t+h}^\ell}\right) = \overline{c_{\ell j}(h)}, \tag{2}$$

therefore

$$\boldsymbol{C}(-h) = \boldsymbol{C}^*(h). \tag{3}$$

In case of a real valued time series, $\boldsymbol{C}(-h) = \boldsymbol{C}^T(h)$.

By the Cauchy–Schwartz inequality, for any $j, k = 1, \dots, d$ and $h \in \mathbb{Z}$,

$$|c_{jk}(h)| = \left|\mathbb{E}\left(X_h^j \overline{X_0^k}\right)\right| \le \left[\mathbb{E}\left|X_h^j\right|^2 \mathbb{E}\left|X_0^k\right|^2\right]^{\frac{1}{2}} = \left[c_{jj}(0)c_{kk}(0)\right]^{\frac{1}{2}},$$

and so, $|c_{jj}(h)| \le c_{jj}(0)$.

While the covariance matrix $\boldsymbol{C}(0)$ is *self-adjoint (Hermitian)* by (3), $\boldsymbol{C}(h)$ with a fixed time lag $h \neq 0$ is not self-adjoint in general. However, for an arbitrary $n \ge 1$, let us consider the following $nd \times nd$ matrix, which is a block Toeplitz matrix:

$$\mathfrak{C}_n := \begin{bmatrix} \boldsymbol{C}(0) & \boldsymbol{C}(1) & \boldsymbol{C}(2) & \cdots & \boldsymbol{C}(n-1) \\ \boldsymbol{C}^*(1) & \boldsymbol{C}(0) & \boldsymbol{C}(1) & \cdots & \boldsymbol{C}(n-2) \\ \boldsymbol{C}^*(2) & \boldsymbol{C}^*(1) & \boldsymbol{C}(0) & \cdots & \boldsymbol{C}(n-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}^*(n-1) & \boldsymbol{C}^*(n-2) & \boldsymbol{C}^*(n-3) & \cdots & \boldsymbol{C}(0) \end{bmatrix}. \tag{4}$$

Its (i,j) block is $\boldsymbol{C}(j-i)$. It is obviously self-adjoint and positive semidefinite as it is the usual covariance matrix of the compounded nd-dimensional random vector $(\mathbf{X}_1^T, \mathbf{X}_2^T, \dots, \mathbf{X}_n^T)^T$.

The next two theorems give important characterizations of weakly stationary time series in general.

Theorem 1. *(a) If a d-dimensional time series* $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ *is weakly stationary then its covariance function* $\mathbf{C}(h)$, $h \in \mathbb{Z}$, *is non-negative definite (positive semidefinite), which means that*

$$\sum_{j,k=1}^{n} \mathbf{a}_k^* \boldsymbol{C}(k-j)\, \mathbf{a}_j \geq 0, \qquad \forall n \geq 1, \quad \forall \mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{C}^d. \tag{5}$$

Equivalently, the matrix $\mathfrak{C}_n$ *in* (4) *is self-adjoint and non-negative definite:*

$$\mathbf{a}^* \mathfrak{C}_n \mathbf{a} \geq 0, \qquad \forall n \geq 1, \quad \forall \mathbf{a} \in \mathbb{C}^{nd}. \tag{6}$$

(b) Conversely, to any non-negative definite matrix function $\mathbf{C}(h)$, $h \in \mathbb{Z}$, *one can find a weakly stationary time series* $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$, *with this covariance function.*

Note that the fact that $\mathbf{C}(h)$, $h \in \mathbb{Z}$, is a positive semidefinite matrix function does not mean that $\boldsymbol{C}(h)$s are positive semidefinite matrices. However, Equation (5) just means that the block Toeplitz matrix $\mathfrak{C}_n$ is positive semidefinite in terms of its blocks.

Theorem 2. *(a) If a d-dimensional time series* $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ *is weakly stationary, then it has a non-negative definite spectral measure matrix* $d\boldsymbol{F}$ *on* $[-\pi, \pi]$ *such that the covariance matrix function* $\boldsymbol{C}(h)$, $h \in \mathbb{Z}$, *of* $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ *can be represented as the Fourier transform of* $d\boldsymbol{F}$*:*

$$\boldsymbol{C}(h) = \int_{-\pi}^{\pi} e^{ih\omega} d\boldsymbol{F}(\omega) \quad (h \in \mathbb{Z}). \tag{7}$$

Note that a measure matrix $d\boldsymbol{F} = [dF^{rs}]_{r,s=1}^d$ *on* $[-\pi, \pi]$ *is called non-negative definite if for any interval* $(\alpha, \beta] \subset [-\pi, \pi]$ *and for any* $z_1, \dots, z_n \in \mathbb{C}$, *it holds that*

$$\sum_{r,s=1}^{d} dF^{rs}((\alpha, \beta]) z_r \overline{z_s} \geq 0. \tag{8}$$

(b) Conversely, to any non-negative definite measure matrix $d\boldsymbol{F}$ *on* $[-\pi, \pi]$ *one can find a non-negative definite function* $\mathbf{C}(h)$, $h \in \mathbb{Z}$, *and so one can find a weakly stationary time series* $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$, *whose spectral measure matrix is* $d\boldsymbol{F}$.

The proof of these two theorems will be given step-by-step in the sequel in this and the subsequent sections. First, it is very simple that the matrix $\mathfrak{C}_n$ in formula (4) is non-negative definite:

$$\sum_{k,j=1}^{n} \mathbf{a}_k^* \boldsymbol{C}(k-j)\, \mathbf{a}_j = \mathbb{E}\left|\sum_{k=1}^{n} \mathbf{a}_k^* \mathbf{X}_k\right|^2 \geq 0 \tag{9}$$

for any $n \geq 1$ and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{C}^d$, see (3) as well. This proves Theorem 1(a). Later, Corollary 6 will show that the converse statement in Theorem 1(b) is also true.

Theorem 3 (Herglotz theorem) below proves Theorem 2(a) in the one-dimensional case. The general case will be established in Corollary 4. Then Corollary 7 will show the converse statement in Theorem 2(b).

Naturally, the one-dimensional cases of formulas (4) and (9) are simpler. If $\{X_t\}_{t\in\mathbb{Z}}$ is a one-dimensional stationary time series and its covariance function is

$$c(h) := \operatorname{Cov}(X_{t+h}, X_t) \quad (t, h \in \mathbb{Z}),$$

then

$$\boldsymbol{C}_n := \begin{bmatrix} c(0) & c(1) & c(2) & \cdots & c(n-1) \\ c(-1) & c(0) & c(1) & \cdots & c(n-2) \\ c(-2) & c(-1) & c(0) & \cdots & c(n-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c(-n+1) & c(-n+2) & c(-n+3) & \cdots & c(0) \end{bmatrix} \tag{10}$$

is an ordinary Toeplitz matrix. Clearly, it is self-adjoint and non-negative definite for any $n \geq 1$:

$$\sum_{j,k=1}^{n} c(k-j)\, z_k\, \overline{z_j} = \mathbb{E}\left|\sum_{k=1}^{n} z_k X_k\right|^2 \geq 0 \quad (z_1, \dots, z_n \in \mathbb{C}). \tag{11}$$

In case of a real time series of zero expectation, this is $\operatorname{Var} \sum_{k=1}^{n} z_k X_k$, and so, it is non-negative. Also, it is the usual covariance matrix of $(X_1, \dots, X_n)^T$ and it is positive semidefinite for this reason too. This implies that in the d-dimensional case, the Toeplitz matrix of the covariance function of each coordinate X_t^j $(j = 1, \dots, d)$ is also self-adjoint and non-negative definite for any $n \geq 1$.

Theorem 3. *(Herglotz theorem) Let $c : \mathbb{Z} \to \mathbb{C}$ be a non-negative definite function. Then c has a spectral representation*

$$c(h) = \int_{-\pi}^{\pi} e^{ih\omega} dF(\omega), \quad h \in \mathbb{Z},$$

where dF is a unique bounded non-negative measure on $[-\pi, \pi]$.

Proof. (can be skipped) Since c is non-negative definite, with any $n \geq 1$ and $z_j = e^{-ij\omega}$, $j = 1, \dots, n$, we get that

$$0 \leq \sum_{j,\ell=1}^{n} c(j-\ell)\, e^{-i(j-\ell)\omega} = \sum_{k=-(n-1)}^{n-1} c(k)\, e^{-ik\omega}\, (n - |k|).$$

Define

$$f_n(\omega) := \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} c(k)\, e^{-ik\omega} \left(1 - \frac{|k|}{n}\right). \tag{12}$$

Then

$$f_n(\omega) \geq 0, \qquad \int_{-\pi}^{\pi} f_n(\omega) d\omega = c(0) \geq 0 \qquad (n \geq 1).$$

Let dF_n be the measure on $[-\pi, \pi]$ whose non-negative density is f_n for $n \geq 1$. Since these measures are bounded and the interval is compact, by Helly's selection theorem, there exists a subsequence $\{n'\}$ such that the sequence of measures $\{dF_{n'}\}$ converges weakly to a limit dF; this will be the desired measure of the theorem.

First, for any integer h such that $|h| \leq m$,

$$\int_{-\pi}^{\pi} e^{ih\omega} f_n(\omega) d\omega = c(h) \left(1 - \frac{|h|}{n}\right).$$

Thus for any $h \in \mathbb{Z}$,

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} e^{ih\omega} f_n(\omega) d\omega = c(h).$$

By the definition of weak convergence, for a suitable subsequence $\{n'\}$ one has

$$\lim_{n' \to \infty} \int_{-\pi}^{\pi} e^{ih\omega} dF_{n'}(\omega) = \int_{-\pi}^{\pi} e^{ih\omega} dF(\omega).$$

Comparing these expressions proves the claimed representation.

The uniqueness of the measure dF follows from the fact that any continuous function on $[-\pi, \pi]$ can be uniformly approximated by trigonometric polynomials (Weierstrass' theorem) and integrals of continuous functions determine the measure by the Riesz representation theorem. □

Later, in Corollary 7, we will see that the converse of Herglotz theorem is also true. The Herglotz theorem implies that each covariance function $c_{jj}(h)$ ($h \in \mathbb{Z}$), for $j = 1, \ldots, d$, has a spectral representation

$$c_{jj}(h) = \int_{-\pi}^{\pi} e^{ih\omega} dF^j(\omega), \quad F^j(-\pi) = 0, \quad F^j(\pi) = c_{jj}(0). \tag{13}$$

Here $F^j(\omega) := dF^j((-\pi, \omega])$, $\omega \in (-\pi, \pi]$, is a non-decreasing function.

Remark 1. *(a) An important special case of the Herglotz theorem is when the covariance function $c(h)$ is absolutely summable :*

$$\sum_{h=-\infty}^{\infty} |c(h)| < \infty,$$

that is, $\{c(h)\}_{h\in\mathbb{Z}}$ is in ℓ^1. Then by (12), *for $k \leq \ell$ and for any $\omega \in [-\pi, \pi]$,*

$$|f_\ell(\omega) - f_k(\omega)| \leq \frac{1}{2\pi} \sum_{|h|\geq k} |c(h)| \to 0$$

as $k \to \infty$. Thus f_k uniformly converges to a continuous spectral density function f *as $k \to \infty$, so dF is absolutely continuous w.r.t. Lebesgue measure, and*

$$c(h) = \int_{-\pi}^{\pi} e^{ih\omega} f(\omega) d\omega, \quad dF(\omega) = f(\omega) d\omega, \tag{14}$$

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} c(h)\, e^{-ih\omega}, \quad f(\omega) \geq 0, \quad \omega \in [-\pi, \pi], \tag{15}$$

the series (15) *being pointwise convergent for $\omega \in [-\pi, \pi]$. Note that in this case f is a continuous function.*

(b) Another important special case is when the covariance function is square summable :

$$\sum_{h=-\infty}^{\infty} |c(h)|^2 < \infty,$$

that is, $\{c(h)\}_{h\in\mathbb{Z}}$ is in ℓ^2. Though this condition is weaker than the absolute summability, formulas (14) *and* (15) *still hold by the Riesz–Fischer theorem. So the spectral measure is absolutely continuous and the corresponding Fourier series converges to the spectral density f, though in this case in the $L^2([-\pi,\pi],\mathcal{B},d\omega)$ sense.*

Remark 2. *The spectrum of a time series with discrete times $\mathbb{Z}$ has been defined on the interval $[-\pi,\pi]$. Another usual approach is to define the spectrum on the unit circle $T=\{z\in\mathbb{C}:|z|=1\}$. Since there is a one-to-one correspondence between the two:*

$$(-\pi,\pi]\ni\omega\leftrightarrow e^{-i\omega}\in T,$$

there exists also a one-to-one correspondence between functions defined on the two. If $\phi:(-\pi,\pi]\to\mathbb{C}$, then there is a unique function $\Phi:T\to\mathbb{C}$ such that $\phi(\omega)=\Phi(e^{-i\omega})$ for any $\omega\in(-\pi,\pi]$.

So $\mathbf{X}_t\to e^{it\cdot}$ and $S\mathbf{X}_t=e^{i\cdot}e^{it\cdot}=e^{i(t+1)\cdot}$ that corresponds to $\mathbf{X}_{t+1}$, where S denotes the right time-shift, to be introduced soon.

2 Spectral representation of multidimensional stationary time series

Let $\mathbf{X}_t=(X_t^1,\dots,X_t^d)$, $t\in\mathbb{Z}$, be a d-dimensional weakly stationary time series. For each $j=1,\dots,d$ fixed, take the set of all finite linear combinations (with complex coefficients) of the random variables X_t^j ($t\in\mathbb{Z}$), and let $H(X^j)$ denote its closure in the Hilbert space $L^2(\Omega,\mathcal{F},\mathbb{P})$. Also, let $H(\mathbf{X})$ denote the closure in $L^2(\Omega,\mathcal{F},\mathbb{P})$ of the linear span of $H(X^1)\cup\dots\cup H(X^d)$:

$$H(\mathbf{X})=\overline{\text{span}}\{\mathbf{X}_t:t\in\mathbb{Z}\}:=\overline{\text{span}}\{X_t^j:t\in\mathbb{Z},\ j=1,\dots,d\}.$$

The right time shift (or forward shift) S is a linear operator given by $SX_t^j=X_{t+1}^j$ for $j=1,\dots,d$. The operator S can be extended to $H(\mathbf{X})$ by linearity and continuity. The inverse S^{-1} of S is *the left time shift (or backward shift)*, defined similarly. Thus we may write that $S^t\mathbf{X}_0=\mathbf{X}_t$ for any $t\in\mathbb{Z}$. Consider the following two equations for any $k,t\in\mathbb{Z}$,

$$\begin{aligned}&\text{Cov}(\mathbf{X}_{t+k},\mathbf{X}_t)=\text{Cov}(S^{t+k}\mathbf{X}_0,S^t\mathbf{X}_0)=\text{Cov}(S^k\mathbf{X}_0,S^{*t}S^t\mathbf{X}_0)\\&\text{Cov}(S^k\mathbf{X}_0,\mathbf{X}_0)=\text{Cov}(\mathbf{X}_k,\mathbf{X}_0).\end{aligned}\tag{16}$$

They are equal to each other if and only if $S^*S = I$, that is, $S^{-1} = S^*$. Thus the right time shift S is a *unitary operator* in $H(\mathbf{X})$ if and only if the time series $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ is weakly stationary.

From now on we assume that $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ is a weakly stationary d-dimensional time series. Then, by continuous extension, S is a unitary operator in each $H(X^j)$ $(j = 1, \ldots, d)$ and the whole $H(\mathbf{X})$ as well. Also, for any $A \subset \mathbb{Z}$ and for any $k \in \mathbb{Z}$,

$$S^k\left(\overline{\mathrm{span}}\{\mathbf{X}_t : t \in A\}\right) = \overline{\mathrm{span}}\{\mathbf{X}_{t+k} : t \in A\}.$$

The spectral representation of the covariance function can be extended to a *spectral representation of a time series* $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ itself. To this end, we introduce a natural isometry. Note that a map ψ from a Hilbert space H onto a Hilbert space G is called an *isometry (isometric isomorphism) between* H *and* G, if we have

$$\langle \psi(X), \psi(Y)\rangle_G = \langle X, Y\rangle_H, \qquad \forall X, Y \in H. \tag{17}$$

First consider a 1D weakly stationary time series $\{X_t\}_{t\in\mathbb{Z}}$ with spectral measure dF. Define the linear map $\psi : H(X) \to L^2([-\pi, \pi], \mathcal{B}, dF)$ for a random variable X_t as

$$\psi(X_t) = \{e^{it\omega} : \omega \in (-\pi, \pi]\}.$$

Here $\mathcal{B}$ denotes the σ-field of Borel sets in $[-\pi, \pi]$. We emphasize that the image of the random variable X_t is the *function* $\omega \mapsto e^{it\omega}$ from $(-\pi, \pi]$ onto the unit circle T. Then ψ is indeed an isometry:

$$\begin{aligned}\langle X_t, X_s\rangle_{H(X)} &= \mathbb{E}\left(X_t\overline{X_s}\right) = c_{jj}(t-s) = \int_{-\pi}^{\pi} e^{i(t-s)\omega} dF(\omega) \\ &= \int_{-\pi}^{\pi} e^{it\omega}e^{-is\omega} dF(\omega) = \langle e^{it\omega}, e^{is\omega}\rangle_{dF}.\end{aligned}$$

Next, this isometry ψ can be extended to finite linear combinations as

$$\psi\left(\sum_{k=1}^{m} a_k X_{t_k}\right) = \sum_{k=1}^{m} a_k e^{it_k\omega}, \quad \omega \in (-\pi, \pi],$$

and finally to the whole Hilbert space $H(X)$ by continuity. The image of $H(X)$ is the closure of the set of trigonometric polynomials, which equals the whole $L^2([-\pi, \pi], \mathcal{B}, dF)$.

Returning to the case of d-dimensional weakly stationary time series $\{\mathbf{X}_t\}$, for any index j and component $\{X_t^j\}$, $j = 1, \ldots, d$, we can define an isometry $\psi^j : H(X^j) \to L^2([-\pi, \pi], \mathcal{B}, dF^j)$ as described above. In this isometric isomorphism, the application of S^k in $H(X^j)$ corresponds to a multiplication with the function $e^{ik\omega}$ in $L^2([-\pi, \pi], \mathcal{B}, dF^j)$. Since the trigonometric polynomials are dense in the latter Hilbert space, for any square-integrable periodic function there exists a unique random variable in $H(X^j)$ by $(\psi^j)^{-1}$. In particular, for any $\omega \in (-\pi, \pi]$ and indicator $\mathbf{1}_{(-\pi,\omega]}$, there exists a unique complex-valued random variable

$$Z_\omega^j := (\psi^j)^{-1}(\mathbf{1}_{(-\pi,\omega]}) \in H(X^j),$$

and we define $Z_{-\pi}^j := 0$. Moreover, for any $B \in \mathcal{B}$, there exists a unique random variable $Z_B^j = (\psi^j)^{-1}(\mathbf{1}_B) \in H(X^j)$.

Then the process $(Z_\omega^j)_{\omega \in (-\pi,\pi]}$ has orthogonal increments. Indeed, if $-\pi \leq a < b \leq c < d \leq \pi$, then

$$\begin{aligned} \mathbb{E}\left((Z_b^j - Z_a^j)\overline{(Z_d^j - Z_c^j)}\right) &= \langle Z_b^j - Z_a^j, Z_d^j - Z_c^j \rangle_{H(X^j)} \\ = \langle \mathbf{1}_{(a,b]}, \mathbf{1}_{(c,d]} \rangle_{dF^j} &= \int_{-\pi}^{\pi} \mathbf{1}_{(a,b]}(\omega) \mathbf{1}_{(c,d]}(\omega) dF^j(\omega) = 0, \end{aligned} \tag{18}$$

likewise,

$$\mathbf{E}(|Z_b^j - Z_a^j|^2) = \int_{-\pi}^{\pi} \mathbf{1}_{(a,b]}(\omega) dF^j(\omega) = F^j(b) - F^j(a). \tag{19}$$

Start with *step functions* (or *simple functions*) with an arbitrary positive integer N,

$$\phi(\omega) = \sum_{r=1}^{N-1} a_r \mathbf{1}_{(\omega_r, \omega_{r+1}]}(\omega) \quad (a_r \in \mathbb{C})$$

in $L^2([-\pi, \pi], \mathcal{B}, dF^j)$, where each ω_r is a continuity point of F^j and $-\pi = \omega_1 < \omega_2 < \cdots < \omega_N = \pi$. Define the *stochastic integral* of a step function by

$$\int_{-\pi}^{\pi} \phi(\omega) dZ_\omega := \sum_{r=1}^{N-1} a_r (Z_{\omega_{r+1}} - Z_{\omega_r}). \tag{20}$$

This establishes an isometry between step functions and random variables of the form (20). Since any $\phi \in L^2([-\pi, \pi], \mathcal{B}, dF^j)$ can be approximated by step functions, isometry ψ^j extends to one between the two Hilbert spaces above, and we get an elementary case of stochastic integration:

$$(\psi^j)^{-1}(\phi) = \int_{-\pi}^{\pi} \phi(\omega) dZ_\omega^j.$$

Use this when $\phi(\omega) = e^{it\omega}$:

$$X_t^j = \int_{-\pi}^{\pi} e^{it\omega} dZ_\omega^j \qquad (j = 1, \dots, d). \tag{21}$$

Corollary 1. *With the notations* $\mathbf{X}_t := (X_t^1, \dots, X_t^d)$ $(t \in \mathbb{Z})$ *and* $\mathbf{Z}_\omega := (Z_\omega^1, \dots, Z_\omega^d)$ $(\omega \in [-\pi, \pi])$*, we can write that*

$$\mathbf{X}_t = \int_{-\pi}^{\pi} e^{it\omega} d\mathbf{Z}_\omega \qquad (t \in \mathbb{Z}). \tag{22}$$

This is the spectral representation of the stationary time series $\{\mathbf{X}_t\}_{t \in \mathbb{Z}}$.

Similarly as in the Herglotz theorem (Theorem 3), one can exhibit spectral representation of the non-diagonal entries of the covariance matrix:

$$c_{j\ell}(h) = \int_{-\pi}^{\pi} e^{ih\omega} dF^{j\ell}(\omega), \quad F^{j\ell}(-\pi) = 0, \quad F^{j\ell}(\pi) = c_{j\ell}(0). \tag{23}$$

The only difference is that here the spectral measure is complex in general.

Let us introduce the *spectral measure matrix* $d\boldsymbol{F} := [dF^{j\ell}]_{d \times d}$. First, it is self-adjoint:

$$c_{j\ell}(h) = \int_{-\pi}^{\pi} e^{ih\omega} dF^{j\ell}(\omega) = \overline{c_{\ell j}(-h)} = \int_{-\pi}^{\pi} e^{ih\omega} \overline{dF^{\ell j}(\omega)} \quad \Rightarrow \quad dF^{j\ell} = \overline{dF^{\ell j}}, \tag{24}$$

since by Weierstrass' theorem the trigonometric polynomials are dense in $C[-\pi, \pi]$, the space of continuous functions over $[-\pi, \pi]$, and thus determine the measure by the Riesz representation theorem. Also, $d\boldsymbol{F}$ is non-negative definite:

$$\sum_{r,s=1}^{m} \mathbf{a}_r^* \, d\boldsymbol{F}(B_r \cap B_s) \, \mathbf{a}_s = \mathbb{E} \left| \sum_{r=1}^{m} \mathbf{a}_r^* \, \mathbf{Z}_{B_r} \right|^2 \geq 0 \tag{25}$$

for any $m \geq 1$ and $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{C}^n$; $B_1, \dots, B_m \in \mathcal{B}$. In particular, definition (8) of non-negative definiteness of a spectral measure matrix holds too.

Corollary 2. *In the special case when each $dF^{j\ell}$ are absolutely continuous w.r.t. Lebesgue measure in $[-\pi,\pi]$, that is, $dF^{j\ell}(\omega) = f^{j\ell}(\omega)\,d\omega$ for $j,\ell = 1,\dots,d$, then it follows that the time series has a* spectral density matrix *$\boldsymbol{f} := [f^{j\ell}]_{d\times d}$, which is self-adjoint and non-negative definite.*

Proof. The self-adjointness of $\boldsymbol{f}$ follows from (24), while the non-negative definiteness of $\boldsymbol{f}$ follows from (25). □

Proposition 1. *Let $\{\mathbf{X}_t\}$ be a d-dimensional weakly stationary time series of complex components. Denoting by $\boldsymbol{C}(h) = [c_{pq}(h)]$ the $d\times d$ autocovariance matrices ($\boldsymbol{C}(-h) = \boldsymbol{C}^*(h)$, $h\in\mathbb{Z}$) in the time domain, assume that their entries are absolutely summable, i.e., $\sum_{h=0}^{\infty}|c_{pq}(h)| < \infty$ for $p,q = 1,\dots,d$. Then the spectral density matrix $\boldsymbol{f}(\omega)$ exists in the frequency domain, and it is defined by*

$$\boldsymbol{f}(\omega) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\boldsymbol{C}(h)e^{-ih\omega}, \quad \omega\in[0,2\pi].$$

In view of Corollary 2 it is always self-adjoint, positive semidefinite. Further, if $\boldsymbol{C}(h)$ is a matrix of real entries $\forall h\in\mathbb{Z}$, then the relation

$$\boldsymbol{f}(-\omega) = \boldsymbol{f}(2\pi-\omega) = \overline{\boldsymbol{f}(\omega)}$$

(with entrywise conjugation) holds $\forall\omega\in[0,2\pi]$.

Proof. We have the following equivalent forms for $2\pi\boldsymbol{f}(\omega)$:

$$\begin{aligned}2\pi\boldsymbol{f}(\omega) &= \sum_{h=-\infty}^{\infty}\boldsymbol{C}(h)e^{-ih\omega} = \boldsymbol{C}(0) + \sum_{h=1}^{\infty}[\boldsymbol{C}(h)e^{-ih\omega} + \boldsymbol{C}^*(h)e^{ih\omega}] \\ &= \boldsymbol{C}(0) + \sum_{h=1}^{\infty}[(\boldsymbol{C}(h)+\boldsymbol{C}^*(h))\cos(h\omega) - i(\boldsymbol{C}(h)-\boldsymbol{C}^*(h))\sin(h\omega)].\end{aligned}$$

The first line shows again that $\boldsymbol{f}(\omega)$ is self-adjoint. The second line shows that whenever $\boldsymbol{C}(h)$ is a real matrix and so, $\boldsymbol{C}^*(h) = \boldsymbol{C}^T(h)$, then $\boldsymbol{C}(h) + \boldsymbol{C}^T(h)$ is symmetric and $\boldsymbol{C}(h) - \boldsymbol{C}^T(h)$ is anti-symmetric with 0 diagonal, but $i(\boldsymbol{C}(h) - \boldsymbol{C}^T(h))$ is self-adjoint. Actually, $\sum_{h=1}^{\infty}(\boldsymbol{C}(h)+\boldsymbol{C}^T(h))\cos(h\omega)$ is the real and $\sum_{h=1}^{\infty}(\boldsymbol{C}(h)-\boldsymbol{C}^T(h))\sin(h\omega)$ is the imaginary part of $2\pi\boldsymbol{f}(\omega)$, the (entrywise) conjugate of wich is $2\pi\boldsymbol{f}(-\omega) = 2\pi\boldsymbol{f}(2\pi-\omega)$. □

Corollary 3. *If $\{X_t\}$ is a 1D weakly stationary, real time series, then $f(\omega) \geq 0$ is real, and $f(-\omega) = f(\omega)$, $\forall \omega \in [0, 2\pi]$. If $\{\mathbf{X}_t\}$ is a d-dimensional weakly stationary time series of real components, then $\boldsymbol{C}(h)$ is a matrix of real entries ($\forall h \in \mathbb{Z}$), so $\boldsymbol{f}(-\omega) = \overline{\boldsymbol{f}(\omega)}$ holds, $\forall \omega \in [0, 2\pi]$. But it is not necessary for a time series to have real components so that $\boldsymbol{C}(h)$ be a real matrix. For example, let $\{\mathbf{Y}_t\}$ be a d-dimensional weakly stationary time series of real components with expectation $\mathbf{0}$. Let $\boldsymbol{\mu} \in \mathbb{C}^d$ be a vector of at least one coordinate containing a non-zero imaginary part. Then the time series $\mathbf{X}_t = \mathbf{Y}_t + \boldsymbol{\mu}$ has at least one complex (not purely real) coordinate, still its autcovarance matrix sequence is the same as that of $\mathbf{Y}_t$, so $\boldsymbol{C}(h)$s have real entries.*

When $\boldsymbol{C}(h)$ has complex entries too, then there is no exact relation between $\boldsymbol{f}(-\omega)$ and $\boldsymbol{f}(\omega)$ in general. Indeed, $\boldsymbol{A} := \sum_{h=1}^{\infty} (\boldsymbol{C}(h) + \boldsymbol{C}^*(h)) \cos(h\omega)$ and $\boldsymbol{B} := \sum_{h=1}^{\infty} (\boldsymbol{C}(h) - \boldsymbol{C}^*(h)) \sin(h\omega)$ are complex matrices, say $\boldsymbol{A} = \boldsymbol{A}_1 + i\boldsymbol{A}_2$, $\boldsymbol{B} = \boldsymbol{B}_1 + i\boldsymbol{B}_2$. Then

$$\boldsymbol{f}(\omega) = \boldsymbol{C}_0 + \boldsymbol{A} + i\boldsymbol{B} = \boldsymbol{C}_0 + (\boldsymbol{A}_1 - \boldsymbol{B}_2) + i(\boldsymbol{A}_2 + \boldsymbol{B}_1),$$

whereas

$$\boldsymbol{f}(-\omega) = \boldsymbol{C}_0 + \boldsymbol{A} - i\boldsymbol{B} = \boldsymbol{C}_0 + (\boldsymbol{A}_1 + \boldsymbol{B}_2) + i(\boldsymbol{A}_2 - \boldsymbol{B}_1),$$

which are not the same, unless $\boldsymbol{B} = \boldsymbol{O}$, but it does not hold in general.

Summarizing, there is no extension from the real to the complex case. Consequently, in the real case, we can confine ourselves to $[0, \pi]$, while is the complex case, the whole $[0, 2\pi]$ or $[-\pi, \pi]$ should be used.

Corollary 4. *We can write (23) in matrix form:*

$$\boldsymbol{C}(h) = \int_{-\pi}^{\pi} e^{ih\omega} d\boldsymbol{F}(\omega),$$

or in the case of an absolutely continuous spectral measure:

$$\boldsymbol{C}(h) = \int_{-\pi}^{\pi} e^{ih\omega} \boldsymbol{f}(\omega) d\omega. \tag{26}$$

This proves Theorem 2(a).

Remark 3. *Remark 1 can be extended to the present case as well.*

(a) If for each $j, \ell = 1, \dots, d$ we have

$$\sum_{h=-\infty}^{\infty} |c_{j\ell}(h)| < \infty, \tag{27}$$

that is, $\{\boldsymbol{C}(h)\}_{h\in\mathbb{Z}} \in \ell^1$, then the time series $\{\mathbf{X}_t\}$ has a continuous *spectral density matrix $\boldsymbol{f}(\omega) = [f^{j\ell}(\omega)]_{d\times d}$,*

$$\boldsymbol{C}(h) = \int_{-\pi}^{\pi} e^{ih\omega} \boldsymbol{f}(\omega) d\omega, \tag{28}$$

$$\boldsymbol{f}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \boldsymbol{C}(h)\, e^{-ih\omega}, \quad \omega \in [-\pi, \pi], \tag{29}$$

where the series converges pointwise.

(b) By the Riesz–Fischer theorem, condition (27) *can be weakened as*

$$\sum_{h=-\infty}^{\infty} |c_{j\ell}(h)|^2 < \infty, \tag{30}$$

that is, $\{\boldsymbol{C}(h)\}_{h\in\mathbb{Z}} \in \ell^2$ for every $j, \ell = 1, \dots, d$. Then the series (29) *converges in $L^2([-\pi, \pi], \mathcal{B}, d\omega)$ entrywise, so the spectral density $\boldsymbol{f}(\omega)$ exists and* (28) *holds.*

Corollary 5. *The following relationship between the orthogonal increment process $\mathbf{Z}_\omega$ and the spectral cdf $\mathbf{F}(\omega)$ is valid:*

$$\mathbf{F}(\omega) = \mathbb{E}(\mathbf{Z}_\omega \mathbf{Z}_\omega^*) \quad (\omega \in [-\pi, \pi]).$$

Definition 4. *An important special case of weakly stationary time series is a so-called* **white noise process** *. Given a self-adjoint non-negative definite matrix $\boldsymbol{\Sigma} = [\sigma_{jk}] \in \mathbb{C}^{d\times d}$, the stationary sequence $\{\boldsymbol{\xi}_t\}_{t\in\mathbb{Z}}$ is called a white noise process with covariance matrix $\boldsymbol{\Sigma}$, denoted* $\mathrm{WN}(\boldsymbol{\Sigma})$*, if $\mathbb{E}(\boldsymbol{\xi}_t) = 0$ for all $t \in \mathbb{Z}$ and its autocovariance function is given by*

$$\boldsymbol{C}_{\boldsymbol{\xi}}(h) = \mathbb{E}(\boldsymbol{\xi}_{t+h}\boldsymbol{\xi}_t^*) = \delta_{h0}\boldsymbol{\Sigma} \qquad (h \in \mathbb{Z}), \tag{31}$$

*where $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ if $j \neq k$ (*Kronecker delta*). It means that the values of $\{\boldsymbol{\xi}_t\}$ are orthogonal (uncorrelated) longitudinally for different* *time instants, while its coordinates may be correlated cross-sectionally, that is, at the same time instant.*

The special case $\mathrm{WN}(\boldsymbol{I}_d)$ *is an* **orthonormal sequence**, *whose coordinates are uncorrelated cross-sectionally as well.*

By Remark 3 and (31), $\{\boldsymbol{\xi}_t\} \sim \mathrm{WN}(\boldsymbol{\Sigma})$ *has spectral density* $\boldsymbol{f}_{\boldsymbol{\xi}}(\omega) = [f_{\boldsymbol{\xi}}^{jk}(\omega)]_{d\times d}$*:*

$$f_{\boldsymbol{\xi}}^{jk}(\omega) = \frac{1}{2\pi}\sigma_{jk}, \quad \boldsymbol{f}_{\boldsymbol{\xi}}(\omega) = \frac{1}{2\pi}\boldsymbol{\Sigma}.$$

This explains the name "white noise": the spectral density is the same constant at every frequency ω*, like the spectrum of ideal white light.*

The last section is optional (only for those who are more deeply interested).

3 Constructions of stationary time series

[This section can be skipped. However, you can use these constructions in your homework assignment.]

There are some standard constructions of a stationary time series with a given covariance function or with a given spectral measure.

3.1 Construction 1

Suppose that $\boldsymbol{C} : \mathbb{Z} \to \mathbb{C}^{d\times d}$ is a non-negative definite function:

$$\sum_{j,k=1}^{n} \mathbf{a}_k^* \boldsymbol{C}(k-j)\,\mathbf{a}_j \geq 0, \qquad \forall n \geq 1, \quad \forall \mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{C}^d.$$

Equivalently, the block Toeplitz matrix $\mathfrak{C}_n$ defined by (4) is self-adjoint and non-negative definite for any $n \geq 1$.

We are going to construct a d-dimensional time series $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ with the given covariance function

$$\mathbb{E}(\mathbf{X}_t\mathbf{X}_s^*) = \boldsymbol{C}(t-s), \qquad t, s \in \mathbb{Z}. \tag{32}$$

The construction goes by induction, defining the value of the time series at $t = 0$, then at $t = 1$, then at $t = -1$, then at $t = 2$, then at $t = -2$, and so on. However, the very first thing is to choose an orthonormal sequence of random variables $\{\xi_j\}_{j=0}^{\infty}$ with expectation 0 on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For example, one can choose a sequence of independent tossing of a fair coin $\mathbb{P}(\xi_j = \pm 1) = \frac{1}{2}$ $(j = 0, 1, 2, \dots)$; or, one can choose a sequence $\{\xi_j\}_{j=0}^{\infty}$ of independent standard normal variables with probability density function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad (x \in \mathbb{R}).$$

We define the Hilbert space H that is going to be a basis of the construction as the closed linear span $H := \overline{\text{span}}\{\xi_j : j = 0, 1, 2, \dots\} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$.

At the beginning we set

$$\boldsymbol{C}(0) = \boldsymbol{A}_0 \boldsymbol{A}_0^*, \quad r := \operatorname{rank} \boldsymbol{C}(0), \quad \mathbf{X}_0 := \boldsymbol{A}_0 \boldsymbol{\xi}_r,$$

where $\boldsymbol{\xi}_r := [\xi_0, \dots, \xi_{r-1}]^T$, using the parsimonious Gram-decomposition (**??**) of the Appendix.

Then each step of the induction will consists of two sub-steps. Assuming for example that a sequence $(\mathbf{X}_{-n}, \mathbf{X}_{-n+1}, \dots, \mathbf{X}_0, \dots, \mathbf{X}_{n-1}, \mathbf{X}_n)$ has already been defined for some $n \geq 0$, the first sub-step will take a preliminary sequence

$$\mathfrak{X}_{-n,n+1} := \left[\tilde{\mathbf{X}}_{-n}, \tilde{\mathbf{X}}_{-n+1}, \dots, \tilde{\mathbf{X}}_0, \dots, \tilde{\mathbf{X}}_n, \tilde{\mathbf{X}}_{n+1}\right]^T. \tag{33}$$

After that a second sub-step will result the new vector $\mathbf{X}_{n+1}$ in the constructed time series. Then the construction of a new vector $\mathbf{X}_{-n-1}$ goes similarly, so not detailed.

For $n \geq 0$ fixed, we would like to define the sequence $\mathfrak{X}_{-n,n+1}$ so that its covariance function be $\boldsymbol{C}(h)$, $h \in \mathbb{Z}$:

$$\mathfrak{C}_{-n,n+1} := \mathbb{E}(\mathfrak{X}_{-n,n+1}\mathfrak{X}_{-n,n+1}^*) = \begin{bmatrix} \boldsymbol{C}(0) & \boldsymbol{C}(-1) & \cdots & \boldsymbol{C}(-2n-1) \\ \boldsymbol{C}(1) & \boldsymbol{C}(0) & \cdots & \boldsymbol{C}(-2n) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}(2n+1) & \boldsymbol{C}(2n) & \cdots & \boldsymbol{C}(0) \end{bmatrix}.$$

Clearly, $\mathfrak{C}_{-n,n+1}$ is a self-adjoint, non-negative definite block Toeplitz matrix of rank $r \leq 2n + 2$. Thus by equation (**??**) in the Appendix, it has a parsimonious Gram-decomposition:

$$\mathfrak{C}_{-n,n+1} = \boldsymbol{A}_{-n,n+1} \cdot \boldsymbol{A}_{-n,n+1}^*, \qquad \boldsymbol{A}_{-n,n+1} \in \mathbb{C}^{(2n+2)\times r}.$$

We set

$$\mathfrak{X}_{-n,n+1} := \boldsymbol{A}_{-n,n+1}\boldsymbol{\xi}_r, \qquad \boldsymbol{\xi}_r := [\xi_0, \dots, \xi_{r-1}]^T. \tag{34}$$

Then, really,

$$\mathbb{E}(\mathfrak{X}_{-n,n+1}\mathfrak{X}^*_{-n,n+1}) = \boldsymbol{A}_{-n,n+1}\mathbb{E}(\boldsymbol{\xi}_r\boldsymbol{\xi}_r^*)\boldsymbol{A}^*_{-n,n+1} = \mathfrak{C}_{-n,n+1}.$$

Now comes the second sub-step of the induction. Assume that the sequence $(\mathbf{X}_{-n}, \mathbf{X}_{-n+1}, \dots, \mathbf{X}_0, \dots, \mathbf{X}_{n-1}, \mathbf{X}_n)$ has been already defined for some $n \geq 0$ and has the covariance function (32) for $s, t \in \{-n, \dots, n\}$. We have also defined the preliminary sequence $\mathfrak{X}_{-n,n+1}$ by (33) and (34). Then define the operator T_{2n+1} by

$$T_{2n+1}\tilde{\mathbf{X}}_t = \mathbf{X}_t \qquad (t = -n, \dots, n).$$

By the construction it follows that T_{2n+1} has the following important property:

$$\mathbb{E}\left((T_{2n+1}\tilde{\mathbf{X}}_t)(T_{2n+1}\tilde{\mathbf{X}}_s)^*\right) = \mathbb{E}(\tilde{\mathbf{X}}_t\tilde{\mathbf{X}}_s^*) = \boldsymbol{C}(t-s), \quad s, t \in \{-n, \dots, n\}. \tag{35}$$

By linearity, T_{2n+1} can be extended to an isomorphy between the finite dimensional spaces

$$\tilde{H}_{2n+1} := \mathrm{Span}_d\{\tilde{\mathbf{X}}_t : t = -n, \dots, n\},$$
$$H_{2n+1} := \mathrm{Span}_d\{\mathbf{X}_t : t = -n, \dots, n\},$$

so that we still have

$$\mathbb{E}\left((T_{2n+1}\tilde{\mathbf{X}})(T_{2n+1}\tilde{\mathbf{Y}})^*\right) = \mathbb{E}(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}^*), \quad \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \tilde{H}_{2n+1}. \tag{36}$$

By Lemma 1 below, we may write that

$$\tilde{\mathbf{X}}_{n+1} = \tilde{\mathbf{X}}^-_{n+1} + \tilde{\mathbf{X}}^+_{n+1}, \quad \tilde{\mathbf{X}}^-_{n+1} \in \tilde{H}_{2n+1}, \quad \tilde{\mathbf{X}}^+_{n+1} \perp \tilde{H}_{2n+1}. \tag{37}$$

If $\tilde{\mathbf{X}}^+_{n+1} = \mathbf{0}$, then we are ready: $\mathbf{X}_{n+1} := T_{2n+1}\tilde{\mathbf{X}}_{n+1}$ belongs to the already defined subspace H_{2n+1}. Otherwise, set $\mathbf{U}_{n+1} := \mathbf{X}^+_{n+1}/\|\mathbf{X}^+_{n+1}\|$ and define $\tilde{H}^+_{2n+1} := \mathrm{Span}_d\{\tilde{H}_{2n+1}, \mathbf{U}_{n+1}\}$.

Let ξ_k be the first random variable in the sequence $\{\xi_j\}_{j=0}^{\infty}$ that has not been used so far in the construction of the sequence $(\mathbf{X}_{-n}, \dots, \mathbf{X}_n)$. Define $T_{2n+1}\mathbf{U}_{n+1} = \mathbf{V}_{n+1} := [\xi_k, \xi_{k+1}, \dots, \xi_{k+d-1}]^T/\sqrt{d}$ and $H^+_{2n+1} := \mathrm{Span}_d\{H_{2n+1}, \mathbf{V}_{n+1}\}$.

Extend T_{2n+1} between $\tilde{H}^+_{2n+1}$ and H^+_{2n+1} by linearity, and define $\mathbf{X}_{n+1} := T_{2n+1}\tilde{\mathbf{X}}_{n+1}$. Then by (35) and (37),

$$\mathbb{E}(\mathbf{X}_{n+1}\mathbf{X}_t^*) = \mathbb{E}\left((T_{2n+1}\tilde{\mathbf{X}}_{n+1})(T_{2n+1}\tilde{\mathbf{X}}_t)^*\right) = \mathbb{E}(\tilde{\mathbf{X}}_{n+1}\tilde{\mathbf{X}}_t^*) = \boldsymbol{C}(n+1-t)$$

for any $t = -n, \ldots, n$. This completes the induction.

The next lemma describes a slight generalization of the standard projection theorem in Hilbert spaces; see Appendix C as well.

Lemma 1. *Let $M_d = M \times \cdots \times M$ be a closed subspace in $L^2_d(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbf{Y} \in L^2_d(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a unique $\hat{\mathbf{Y}} \in M_d$ such that*

$$\|\mathbf{Y} - \hat{\mathbf{Y}}\| \le \|\mathbf{Y} - \mathbf{Z}\| \text{ for any } \mathbf{Z} \in M_d, \tag{38}$$

equivalently,

$$\mathrm{Cov}(\mathbf{Y} - \hat{\mathbf{Y}}, \mathbf{Z}) = \boldsymbol{O} \text{ for any } \mathbf{Z} \in M_d, \quad (\mathbf{Y} - \hat{\mathbf{Y}}) \perp M_d. \tag{39}$$

For this $\hat{\mathbf{Y}}$, we have $\hat{Y}^j = \mathrm{Proj}_M Y^j$ $(j = 1, \ldots, d)$, the standard orthogonal projection of Y^j to the closed Hilbert subspace $M \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. Define $\hat{\mathbf{Y}} = (\hat{Y}^1, \ldots, \hat{Y}^d)$ componentwise by the standard projection theorem:

$$\hat{Y}^j := \mathrm{Proj}_M Y^j, \qquad j = 1, \ldots, d.$$

Then for arbitrary $\mathbf{Z} \in M_d$ let $\mathrm{Cov}(\mathbf{Y} - \hat{\mathbf{Y}}, \mathbf{Z}) =: [\gamma_{jk}]^d_{j,k=1}$, for which we have

$$\gamma_{jk} = \langle Y^j - \hat{Y}^j, Z^k \rangle = 0 \quad \forall j, k.$$

This proves (39).

Also,

$$\|\mathbf{Y} - \mathbf{Z}\|^2 = \sum_{j=1}^d \|Y^j - Z^j\|^2 = \sum_{j=1}^d \left\{ \|Y^j - \hat{Y}^j\|^2 + \|\hat{Y}^j - Z^j\|^2 \right\}, \tag{40}$$

which proves (38).

Conversely, if we choose any $\mathbf{Z} \in M_d$, $\mathbf{Z} \neq \hat{\mathbf{Y}}$, then (40) shows that it cannot have the minimum property (38). □

Corollary 6. *Formula* (9) *shows that the covariance function of any stationary time series is non-negative definite. Conversely, Construction 1 proves that for any non-negative definite function $\boldsymbol{C}(h)$, $h \in \mathbb{Z}$, one can construct a stationary time series with this covariance function.*

3.2 Construction 2

Assume that we are given a $d \times d$ matrix $d\boldsymbol{F} = [dF^{rs}]_{r,s=1}^{d}$ whose entries are finite complex measures on $([-\pi, \pi], \mathcal{B})$ and which is non-negative definite:

$$\sum_{r,s=1}^{d} dF^{rs}((\alpha,\beta])z_r\bar{z}_s = \sum_{r,s=1}^{d} \int_{-\pi}^{\pi} z_r\bar{z}_s \, \mathbf{1}_{(\alpha,\beta]}(\omega)\, dF^{rs}(\omega) \geq 0, \tag{41}$$

for any interval $(\alpha, \beta] \subset [-\pi, \pi]$ and $z_1, \ldots, z_d \in \mathbb{C}$. Equivalently, the matrix

$$\Delta_{\alpha\beta}\boldsymbol{F} := [\Delta_{\alpha\beta}F^{rs}]_{r,s=1}^{d} := [F^{rs}(\beta) - F^{rs}(\alpha)]_{r,s=1}^{d}$$

is non-negative definite for any $(\alpha, \beta] \subset [-\pi, \pi]$, that is,

$$\sum_{r,s=1}^{d} \Delta_{\alpha\beta}F^{rs}z_r\bar{z}_s \geq 0. \tag{42}$$

We would like to construct a time series whose spectral measure matrix is the given $d\boldsymbol{F}$. One way to do it is to show that $d\boldsymbol{F}$ defines a non-negative definite function $\boldsymbol{C}(h)$, $h \in \mathbb{Z}$, and then using Construction 1 to complete the construction.

It should be noted that inequality (41) can be extended to sums of the

$$\sum_{r,s=1}^{d} \int_{-\pi}^{\pi} \sum_{j=1}^{n} z_r(j)\bar{z}_s(j)\, \mathbf{1}_{(\alpha_j,\beta_j]}(\omega)\, dF^{rs}(\omega) \geq 0, \quad n \geq 1, \tag{43}$$

where $(\alpha_j, \beta_j] \subset [-\pi, \pi]$ and $z_1(j), \ldots, z_d(j) \in \mathbb{C}$ for any $n \geq 1$ and $j = 1, \ldots, n$. It is clear that the class of step-functions

$$g(\omega) := \sum_{j=1}^{n} z(j)\, \mathbf{1}_{(\alpha_j,\beta_j]}(\omega), \quad n \geq 1, \quad \omega \in [-\pi, \pi],$$

is dense in $L^2([-\pi, \pi], \mathcal{B}, \operatorname{tr}(d\boldsymbol{F}(\omega)))$, where $\operatorname{tr}(d\boldsymbol{F}(\omega))$ denotes the trace of $d\boldsymbol{F}$, which dominates any measure entry in $d\boldsymbol{F}$. Thus one can extend inequality (43) to the case

$$\sum_{r,s=1}^{d} \int_{-\pi}^{\pi} g_r(\omega)\bar{g}_s(\omega)\, dF^{rs}(\omega) \geq 0, \tag{44}$$

where $g_1, \dots, g_d \in L^2([-\pi, \pi], \mathcal{B}, \operatorname{tr}(d\boldsymbol{F}(\omega)))$.

Define

$$\boldsymbol{C}(h) := \int_{-\pi}^{\pi} e^{ih\omega} d\boldsymbol{F}(\omega) \in \mathbb{C}^{d\times d}, \qquad h \in \mathbb{Z}.$$

Take an arbitrary integer $n \geq 1$ and arbitrary vectors $\mathbf{a}_k = (a_k^1, \dots, a_k^d) \in \mathbb{C}^d$ for $k = 1, \dots, n$. Define the trigonometric polynomials

$$\zeta_r(\omega) := \sum_{k=1}^{n} a_k^r \, e^{-ik\omega}, \qquad r = 1, \dots, d.$$

Then by inequality (44) we have

$$\sum_{j,k=1}^{n} \mathbf{a}_k^* \boldsymbol{C}(k-j)\, \mathbf{a}_j = \sum_{j,k=1}^{n} \mathbf{a}_k^* \int_{-\pi}^{\pi} e^{i(k-j)\omega} d\boldsymbol{F}(\omega)\, \mathbf{a}_j$$

$$= \sum_{r,s=1}^{d} \int_{-\pi}^{\pi} \zeta_r(\omega) \bar{\zeta}_s(\omega) dF^{rs}(\omega) \geq 0.$$

Thus the matrix function $\boldsymbol{C}(h)$, $h \in \mathbb{Z}$, is non-negative definite, so by Construction 1 a stationary time series can be constructed with this covariance matrix function.

Corollary 7. *Equation* (25) *shows that the spectral measure matrix of any stationary time series is non-negative definite. Conversely, Construction 2 proves that for any non-negative definite measure matrix one can construct a stationary time series with this spectral measure.*

If $d = 1$, then (42) *holds if and only if F is a right continuous, non-decreasing function on $[-\pi, \pi]$ such that $F(-\pi) = 0$, $F(\pi) < \infty$. It implies the converse of Herglotz theorem, Theorem 3. For any such distribution function F, its Fourier transform*

$$c(r) = \int_{-\pi}^{\pi} e^{ir\omega} dF(\omega) \quad (r \in \mathbb{Z})$$

defines a non-negative definite function c.

If $d = 2$, then (42) *holds if and only if for any $-\pi \leq \alpha \leq \beta \leq \pi$,*

$$\Delta_{\alpha\beta} F^{rr} \geq 0 \quad (r = 1, 2) \text{ and } \begin{vmatrix} \Delta_{\alpha\beta} F^{11} & \Delta_{\alpha\beta} F^{12} \\ \Delta_{\alpha\beta} F^{21} & \Delta_{\alpha\beta} F^{22} \end{vmatrix} \geq 0.$$

3.3 Construction 3

Let us assume that the d-dimensional stationary time series $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ we would like to construct has an absolutely continuous spectral measure with given density matrix $\boldsymbol{f}$ (which is a self-adjoint, non-negative definite matrix valued function) and suppose that $\boldsymbol{f}(\omega)$ has constant rank $r \leq d$ for a.e. $\omega \in [-\pi, \pi]$. Then we can take the parsimonious Gram-decomposition (**??**) of $2\pi\boldsymbol{f}$:

$$\boldsymbol{f}(\omega) = \frac{1}{2\pi}\boldsymbol{\phi}(\omega)\boldsymbol{\phi}^*(\omega), \qquad \boldsymbol{\phi}(\omega) \in \mathbb{C}^{d\times r},$$

for a.e. $\omega \in [-\pi, \pi]$. (Compare with Theorem **??**.)

Then we may define a d-dimensional stationary Gaussian time series $\{\mathbf{X}_t\}$ with spectral density $\boldsymbol{f}$ using Itô's stochastic integration. Let $\mathbf{B}(\omega)$ be a standard r-dimensional Brownian motion (Wiener process) on the interval $[-\pi, \pi]$. Define a time series as

$$\mathbf{X}_t := \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{it\omega}\boldsymbol{\phi}(\omega)d\mathbf{B}(\omega), \qquad t \in \mathbb{Z}. \tag{45}$$

It is well-known that then $\{\mathbf{X}_t\}$ is a Gaussian process, $\mathbb{E}\mathbf{X}_t = \mathbf{0}$ for any t, and by Itô isometry, the covariance function is

$$\boldsymbol{C}(h) = \mathbb{E}(\mathbf{X}_{t+h}\mathbf{X}_t^*) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(t+h)\omega}\boldsymbol{\phi}(\omega)e^{-it\omega}\boldsymbol{\phi}^*(\omega)d\omega = \int_{-\pi}^{\pi} e^{ih\omega}\boldsymbol{f}(\omega)d\omega$$

for any $h \in \mathbb{Z}$. Thus this time series is stationary with spectral density $\boldsymbol{f}$: this proves the correctness of the construction. In practice one would approximate the stochastic integral in (45) by a stochastic sum.

3.4 Construction 4

3.4.1 Discrete Fourier Transform

First let us review the *Discrete Fourier Transform (DFT)* in a way that conforms our previous setting. For simplicity, choose a positive *odd* integer $2N+1$ and define $\Delta\omega := \frac{2\pi}{2N+1}$. Suppose that the spectral measure of the investigated d-dimensional stationary time series $\{\mathbf{X}_t\}$ is absolutely continuous with density matrix function $\boldsymbol{f}$. We assume that $\boldsymbol{f}$, or an estimate of it, is given at the discrete points $\omega_j := j\Delta\omega \in [-\pi, \pi]$, $j = -N, \dots, N$, called *Fourier frequencies*. Then the DFT of $\boldsymbol{f}$ is defined as

$$\hat{\boldsymbol{C}}(k) = \Delta\omega \sum_{j=-N}^{N} \boldsymbol{f}(\omega_j) e^{ik\omega_j}, \quad k = -N, \dots, N. \tag{46}$$

This finite sequence is a natural estimate of the covariance matrix function, see (26),

$$\boldsymbol{C}(k) = \int_{-\pi}^{\pi} e^{ik\omega} \boldsymbol{f}(\omega) d\omega \qquad (k \in \mathbb{Z}),$$

if $\boldsymbol{f}$ is Riemann integrable and N is large enough. A property of DFT is that it is periodic with period $2N+1$: $\hat{\boldsymbol{C}}(k + 2N + 1) = \hat{\boldsymbol{C}}(k)$ for any k.

Conversely, assume that the covariance matrix function $\boldsymbol{C}(k)$, $k \in \mathbb{Z}$, or an estimate of it, is given and we would like to find an estimate of the spectral density $\boldsymbol{f}$. Then the *inverse DFT (IDFT)* is defined by

$$\hat{\boldsymbol{f}}(\omega_j) = \frac{1}{2\pi} \sum_{k=-N}^{N} \boldsymbol{C}(k)\, e^{-ik\omega_j}, \quad j = -N, \dots, N. \tag{47}$$

It is a natural estimate of the spectral density matrix, see (29):

$$\boldsymbol{f}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \boldsymbol{C}(k) e^{-ik\omega}, \quad \omega \in [-\pi, \pi],$$

if the entries of $\boldsymbol{C}(k)$ are negligible for $|k| > N$. If the entries of $\boldsymbol{C}$ are absolute summable: $\boldsymbol{C} \in \ell^1$, and N is large enough, then this condition holds. A property of IDFT is that it is also periodic with period $2N+1$: $\hat{\boldsymbol{f}}(\omega_{j+2N+1}) = \hat{\boldsymbol{f}}(\omega_j)$ for any j.

If the chosen positive integer is *even*: $2N$, then everything goes similarly as above, except that the indices run from $-N+1$ to N.

It is well-known that $\{e^{ik\omega}\}_{k\in\mathbb{Z}}$ is an orthonormal sequence of functions in $L^2([0, 2\pi], \mathcal{B}, d\omega)$:

$$\langle e^{ik\omega}, e^{i\ell\omega} \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\omega} \overline{e^{i\ell\omega}} d\omega = \delta_{k\ell} \quad (k, \ell \in \mathbb{Z}).$$

Similarly, $\{j \mapsto e^{ik\omega_j}\}_{\{k=-N,\ldots,N\}}$ is an orthonormal sequence of functions on the discrete set of points $\{\omega_j : j = -N, \ldots, N\}$ in the following sense:

$$\begin{aligned}\langle j \mapsto e^{ik\omega_j}, j \mapsto e^{i\ell\omega_j}\rangle &:= \frac{1}{2N+1}\sum_{j=-N}^{N} e^{ik\omega_j}\,\overline{e^{i\ell\omega_j}} \\ &= \frac{1}{2N+1}\sum_{j=-N}^{N} e^{i(k-\ell)\omega_j} = \delta_{k\ell}, \qquad (48)\end{aligned}$$

for any $k, \ell \in \mathbb{Z}$. The next proposition shows that this property implies that the IDFT is really the inverse transformation of the DFT.

Proposition 2. *Assume that $\hat{\boldsymbol{C}}(k)$, $k = -N, \ldots, N$, is the DFT of an approximate spectral density $\hat{\boldsymbol{f}}$ as defined by* (46)*. Then the inverse DFT defined by* (47) *gives*

$$\begin{aligned}\frac{1}{2\pi}\sum_{k=-N}^{N} \hat{\boldsymbol{C}}(k)e^{-ik\omega_j} &= \frac{1}{2\pi}\sum_{k=-N}^{N} e^{-ik\omega_j}\Delta\omega \sum_{\ell=-N}^{N} \hat{\boldsymbol{f}}(\omega_\ell)e^{ik\omega_\ell} \\ &= \sum_{\ell=-N}^{N} \hat{\boldsymbol{f}}(\omega_\ell)\frac{1}{2N+1}\sum_{k=-N}^{N} e^{i(\ell-j)k\Delta\omega} = \hat{\boldsymbol{f}}(\omega_j),\end{aligned}$$

for $j = -N, \ldots, N$.

Similarly, assume that $\hat{\boldsymbol{f}}(\omega_j)$, $j = -N, \ldots, N$, is the inverse DFT of an approximate covariance matrix function $\hat{\boldsymbol{C}}$ as defined by (47)*. Then the DFT defined by* (46) *gives*

$$\begin{aligned}\Delta\omega \sum_{j=-N}^{N} \hat{\boldsymbol{f}}(\omega_j)e^{ik\omega_j} &= \Delta\omega \sum_{j=-N}^{N} e^{ik\omega_j}\frac{1}{2\pi}\sum_{\ell=-N}^{N} \hat{\boldsymbol{C}}(\ell)e^{-i\ell\omega_j} \\ &= \sum_{\ell=-N}^{N} \hat{\boldsymbol{C}}(\ell)\frac{1}{2N+1}\sum_{j=-N}^{N} e^{i(k-\ell)\omega_j} = \hat{\boldsymbol{C}}(k),\end{aligned}$$

for $k = -N, \ldots, N$.

The DFT and IDFT are efficient from an algorithmic point of view, because when $N = 2^n$, they can be evaluated in $O(N \log N)$ steps using Fast Fourier Transform (FFT).

3.4.2 The construction

Like in Construction 3, let us assume that the d-dimensional stationary time series $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ we would like to construct has an absolutely continuous spectral measure with given density matrix $\boldsymbol{f}$ (which is a self-adjoint, non-negative definite matrix valued function) and suppose that $\boldsymbol{f}(\omega)$ has constant rank $r \le d$ for a.e. $\omega \in [-\pi, \pi]$ and is Riemann integrable on $[-\pi, \pi]$. Take the parsimonious Gram-decomposition (**??**) of $2\pi\boldsymbol{f}$:

$$\boldsymbol{f}(\omega) = \frac{1}{2\pi}\boldsymbol{\phi}(\omega)\boldsymbol{\phi}^*(\omega), \qquad \boldsymbol{\phi}(\omega) \in \mathbb{C}^{d\times r},$$

for a.e. $\omega \in [-\pi, \pi]$. (Compare with Theorem **??**.)

The basis of the construction is the spectral representation (22) of $\{\mathbf{X}_t\}$:

$$\mathbf{X}_t = \int_{-\pi}^{\pi} e^{it\omega} d\mathbf{Z}_\omega \quad (t \in \mathbb{Z}), \tag{49}$$

where $\{\mathbf{Z}_\omega\}_{\omega\in[-\pi,\pi]}$ is a d-dimensional process with orthogonal increments. Like above, assume that we have chosen an odd positive integer $2N+1$, $\Delta\omega = \frac{2\pi}{2N+1}$, and the Fourier frequencies $\omega_j = j\Delta\omega$, $j = -N, \dots, N$. Define

$$\Delta\mathbf{Z}(\omega_j) := (2N+1)^{-1/2}\boldsymbol{\phi}(\omega_j)\mathbf{V}_j, \qquad j = -N, \dots, N,$$

where

$$\mathbf{V}_j := [e^{iU_j^1}, \dots, e^{iU_j^r}]^T,$$

and $\{U_j^k : k = 1, \dots, r; j = -N, \dots, N\}$ are independent random variables, uniformly distributed on $[-\pi, \pi]$. Here $\Delta\mathbf{Z}(\omega_j)$ gives a random vector measure of the interval $[\omega_j, \omega_{j+1}]$. It is an increment of a process with orthogonal increments, see (18) and (19):

$$\begin{aligned}\mathbb{E}\left(\Delta\mathbf{Z}(\omega_j)\Delta\mathbf{Z}^*(\omega_\ell)\right) &= \Delta\omega\frac{1}{2\pi}\boldsymbol{\phi}(\omega_j)\mathbb{E}(\mathbf{V}_j\mathbf{V}_\ell^*)\boldsymbol{\phi}(\omega_\ell) \\ &= \delta_{j\ell}\Delta\omega\boldsymbol{f}(\omega_j),\end{aligned} \tag{50}$$

since

$$\mathbb{E}\left(e^{iU_j^k}e^{-iU_\ell^m}\right) = \delta_{j\ell}\delta_{km}, \qquad \mathbb{E}(\mathbf{V}_j\mathbf{V}_\ell^*) = \delta_{j\ell}I_r.$$

As an approximation of (49), for $t = 0, \dots, 2N$ define

$$\hat{\mathbf{X}}_t := \sum_{j=-N}^{N} e^{it\omega_j}\ \Delta\mathbf{Z}\left(\omega_j\right) = \frac{1}{\sqrt{2\pi}}\sum_{j=-N}^{N} e^{it\omega_j}\ \boldsymbol{\phi}(\omega_j)\mathbf{V}_j\sqrt{\Delta\omega}, \tag{51}$$

which is a periodic sequence with period $2N+1$. It is of the form of DFT (46) with coefficients $(\Delta\omega)^{-1}\Delta\mathbf{Z}(\omega_j)$. Compare also the last expression in (51) with construction (45).

By (50) and (51), the covariance matrix function of $\{\hat{\mathbf{X}}_t\}$ is

$$\mathrm{Cov}(\hat{\mathbf{X}}_{t+h}, \hat{\mathbf{X}}_t) = \mathbb{E}(\hat{\mathbf{X}}_{t+h}\hat{\mathbf{X}}_t^*) = \Delta\omega \sum_{j=-N}^{N} \boldsymbol{f}(\omega_j)\, e^{ih\omega_j} = \hat{\boldsymbol{C}}(h).$$

By (46) and Proposition 2 it follows that the spectral density of the sequence $\{\hat{\mathbf{X}}_t\}$ is exactly the given $\boldsymbol{f}(\omega_j)$ at the Fourier frequencies $\{\omega_j : j = -N, \ldots, N\}$.

4 Summary

The d-dimensional, complex-valued, weakly stationary time series $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ can be uniquely characterized by its first and second moments that do not depend on time shift:

$$\boldsymbol{\mu} = \mathbb{E}\mathbf{X}_t, \quad \boldsymbol{C}(h) = \mathbb{E}\left(\mathbf{X}_{t+h} - \boldsymbol{\mu}\right)\left(\mathbf{X}_t - \boldsymbol{\mu}\right)^*, \quad h \in \mathbb{Z},$$

where $\boldsymbol{C}(h)$ is called (auto)covariance matrix function. We usually assume that $\boldsymbol{\mu} = \mathbf{0}$. Note that $\boldsymbol{C}(-h) = \boldsymbol{C}^*(h)$, $h \in \mathbb{Z}$. More generally, we speak of second order processes whenever assume that the above first and second moments determine the process. The following are equivalent to the fact that $\{\mathbf{X}_t\}$ is weakly stationary:

- $\boldsymbol{C}(h)$ is non-negative definite, i.e., $\sum_{k,r=1}^{n} \mathbf{a}_k^* \boldsymbol{C}(k-r)\,\mathbf{a}_r \geq 0$ for $n \geq 1$ and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{C}^d$. Equivalently, the self-adjoint matrix

$$\mathfrak{C}_n := \begin{bmatrix} \boldsymbol{C}(0) & \boldsymbol{C}(1) & \boldsymbol{C}(2) & \cdots & \boldsymbol{C}(n-1) \\ \boldsymbol{C}^*(1) & \boldsymbol{C}(0) & \boldsymbol{C}(1) & \cdots & \boldsymbol{C}(n-2) \\ \boldsymbol{C}^*(2) & \boldsymbol{C}^*(1) & \boldsymbol{C}(0) & \cdots & \boldsymbol{C}(n-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}^*(n-1) & \boldsymbol{C}^*(n-2) & \boldsymbol{C}^*(n-3) & \cdots & \boldsymbol{C}(0) \end{bmatrix},$$

  is positive semidefinite ($\mathfrak{C}_n$ is a block Toeplitz matrix).

- It has a non-negative definite spectral measure matrix $d\boldsymbol{F}$ on $[-\pi, \pi]$ such that its autocovariance matrix function can be represented as the Fourier transform of $d\boldsymbol{F}$: $\boldsymbol{C}(h) = \int_{-\pi}^{\pi} e^{ih\omega} d\boldsymbol{F}(\omega)$, $h \in \mathbb{Z}$.

When each dF^{jk} is absolutely continuous w.r.t. the Lebesgue measure in $[-\pi, \pi]$, that is, $dF^{jk}(\omega) = f^{jk}(\omega)\,d\omega$ for $j, k = 1, \dots, d$, then there exists the $d \times d$ spectral density matrix $\boldsymbol{f} = [f^{jk}]$. A sufficient condition for this is that the entries of $\boldsymbol{C}$ are absolutely summable. The matrix $\boldsymbol{f}(\omega)$ is self-adjoint and non-negative semidefinite for $\omega \in [-\pi, \pi]$; further, when the state space is $\mathbb{R}^d$, then $\boldsymbol{f}(-\omega) = \overline{\boldsymbol{f}(\omega)}$. Also, in the case of absolutely summable $\boldsymbol{C}$,

$$\boldsymbol{C}(h) = \int_{-\pi}^{\pi} e^{ih\omega} \boldsymbol{f}(\omega)\,d\omega \quad \Longleftrightarrow \quad \boldsymbol{f}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \boldsymbol{C}(h) e^{-ih\omega},$$

where $-\pi \leq \omega \leq \pi$. $\boldsymbol{F}$ is sometimes called spectral distribution matrix, while $\boldsymbol{f}$ spectral density matrix. The diagonal entries of $\boldsymbol{f}$ are real functions, whereas the offdiagonal ones can be complex. If we write them in polar form, then we get the so-called amplitude and phase spectrum, respectively.

The weakly stationary time series itself can be represented (with probability one) as

$$\mathbf{X}_t = \int_{-\pi}^{\pi} e^{it\omega} d\mathbf{Z}_\omega, \quad t \in \mathbb{Z}$$

with orthogonal increment process $\mathbf{Z}_\omega$, where $\boldsymbol{F}(\omega) = \mathbb{E}(\mathbf{Z}_\omega \mathbf{Z}_\omega^*)$, $\omega \in [-\pi, \pi]$. This is sometimes called Cramér representation. Note that this generalizes the case of the superposition of sinusoids, where the process has point spectrum. More generally, if the time is continuous, then a spectral measure matrix $d\boldsymbol{F}$ can also be defined on the whole $\mathbb{R}$.