Lesson 11: Non-stationary processes, ARCH, GARCH models, financial applications

Marianna Bolla, BME Math. Inst. and
László Gerencsér, SZTAKI

December 2, 2020

Many financial time series exhibit non-stationary behavior due to, for example, tends and seasonality. There are algorithms at hand which deprive the data of trend and seasonality. Then we transform the time series so that an ARMA or a more sophisticated model can be fitted. We discuss some basic types of these models, mainly used in finance. We follow the description of [1, 2].

1 Harmonic regression

To find a linear or polynomial trend in a time series, regression analysis techniques are used. More interesting is to find seasonal components and periods of a time series. We learned that if ω^* is the place of the first local maximum of the spectral density function, then it has a period approximately $p = [2\pi/\omega^*]$, i.e., $X_{t-p} = X_t$. (This is the shortest period. Longer periods can be found by considering further local maxima.) In this case, the time series can be approximated by sum of harmonics (sine waves):

$$a_0 + \sum_{j=1}^{k} [a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)],$$

where $a_0, a_1, \ldots, a_k, b_1, \ldots, b_k$ are unknown parameters, $\lambda_1, \ldots, \lambda_k$ are fixed frequencies, each being some integer multiple of $2\pi/p$. The number k of the new Fourier frequencies should also be specified. Usually if the above ω^* is close to the Fourier frequency $j^* \frac{2\pi}{n}$ (n is the sample size), then $p \approx \frac{2\pi}{\omega^*} \approx \frac{n}{j^*}$. In this case, we can choose $\lambda_1 = \omega^*$, and some (up to k) multiples of it.

We know that the above approximation is a Type (0) singular process and has discrete spectrum, though it can capture the substantial part of the continuous spectrum. This is also related to signal processing.

2 Integration, cointegration

Let $\{X_t\}$ be a 1D time series which is not even weakly stationary. We are looking for a nonnegative integer d such that the process

$$Y_t := (1-L)^d X_t$$

is already weakly stationary, where L denotes the left (backward) shift operator. In this case X_t becomes *integrated*. The smallest nonnegative integer d for which the process $\{X_t\}$ becomes integrated is called the order of its integration.

In particular, if $d = 0$, then the process itself is weakly stationary. If $d = 1$, then the process of differences $Y_t = X_t - X_{t-1}$ is weakly stationary. It is also applicable to eliminate linear trend.

Integration makes rise to the following generalization of ARMA processes.

Definition 1. *Let d be a nonnegative integer. Then the 1D $\{X_t\}$ is an ARIMA(p, d, q) process if $Y_t = (1-L)^d X_t$ if a causal ARMA(p, q) process.*

This means that $\{X_t\}$ satisfies the difference equation

$$\alpha(L)(1-L)^d X_t = \beta(L) Z_t,$$

where the AR polynomial $\alpha(z)$ has no roots on the closed unit disc (stability), $\beta(z)$ is the MA polynomial, and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Note that for $d \geq 1$ we can add an arbitrary polynomial trend of degree $d-1$ to $\{X_t\}$ without violating the above difference equation. Also, if $d = 1$, then the ARIMA model is equivalent to an ARMA model for the differences $X_t - X_{t-1}$.

ARIMA models are appropriate for slowly decaying positive sample autocorrelation functions (long memory models). A root near 1 of the AR polynomial suggest that the data should be differenced before applying the ARMA model.

Definition 1 can be extended to multi-dimensional processes, and a notion of joint integration (cointegration) can also be defined.

Definition 2. *Let $\{\mathbf{X}_t\}$ be a multi-dimensional real valued process such that it is integrated with order d. It means that $(I-L)^d X_t$ is weakly stationary, but $(I-L)^{d-1} X_t$ is nonstationary, where I is the identity. Then the process is called cointegrated (after the author Granger) if there is a linear combination $\{\boldsymbol{\alpha}^T \mathbf{X}_t\}$ of its components which is integrated of order less than d.*

3 Financial time series

In finance, asset prices and option pricing were widely investigated by Black and Scholes (1973) who later obtained the Nobel-price for it. Their findings are applicable to automated trading too. Without going into details, we introduce some important notions and so-called stylized features of financial time series.

The closing price on trading day t of a particular stock or stock-price index or price of a foreign currency is denoted by P_t. The process $\{P_t\}$ is usually not stationary; however, Louis Bachelier (father of modern financial mathematics) guessed that the price process is a stochastic process with stationary, independent, Gaussian increments. This would imply that prices can take on negative values. Instead, Paul Samuelson defined the *log asset price* $X_t = \log P_t$. It has observed sample-paths, like those of a random walk with stationary, uncorrelated increments. (We learned from Homework Exercise 3 that the random walk itself is not even weakly stationary.) Therefore, we consider the differenced log asset price:

$$Z_t = X_t - X_{t-1} = \log \frac{P_t}{P_{t-1}} = \log \left(1 + \frac{P_t - P_{t-1}}{P_{t-1}} \right) \approx \frac{P_t - P_{t-1}}{P_{t-1}}$$

which is called *log return (or simply return) for day t*. This is close to the *relative return* $\frac{P_t - P_{t-1}}{P_{t-1}}$ if the price does not change much from one day to the next one, relatively to the previous price.

The log return has sample-paths resembling those of white noise; though, there is a strong evidence, that it is not an independent white noise. Much of the analysis of financial time series is devoted to representing and exploiting this dependence which is not visible in the sample autocorrelation function of the process $\{Z_t\}$. The continuous time analogue of a random walk with i.i.d. increments is known as Lévy process, the most familiar examples of which are the Poisson processes and Brownian motion.

If we assume stationarity, e.g., ARMA model for the series $\{Z_t\}$, the stochastic volatily h_t, that is the conditional variance of Z_t given its past values, is independent of t and of $\{Z_s : s < t\}$. However, this property does not always hold. Therefore, the following models were defined by incorporating the stochastic volatility into the model.

The fundamental idea of the ARCH(p) (autoregressive conditional heteroscedastic) model is that

$$Z_t = \sqrt{h_t} e_t, \quad \{e_t\} \sim \text{i.i.d. } \mathcal{N}(0,1)$$

where the *volatility* h_t is related to the past values of Z_t^2 via

$$h_t = a_0 + \sum_{j=1}^{p} a_j Z_{t-j}^2$$

for some positive integer p and parameters $a_0 > 0$, $a_1, \dots, a_p \geq 0$.

The GARCH(p, q) (generalized ARCH) postulates a more general relation:

$$h_t = a_0 + \sum_{j=1}^{p} a_j Z_{t-j}^2 + \sum_{j=1}^{q} b_j h_{t-j},$$

with $b_j \geq 0$, $j = 1, \dots, q$.

These models have been studied intensively together with parameter estimation. They specify a suitable feedback mechanism defining h_t that would ensure that extreme values of the returns generate more activity in the market, expressed in higher volatility, which in turn would explain the phenomenon of volatility clustering. This means that long periods of low volatility are followed by short periods of high volatility.

References

[1] Brockwell, P. J. and Davis, R. A., Introduction to time series and forecasting, 2016, Springer.

[2] Gerencsér, L., Vágó, Zs. and Gerencsér, B., Financial time series, 2013, Pázmány Péter Catholic University, ISBN 978-963-308-161-7.