Dynamic Factor Analysis

Since Geweke (1977) in [8] generalized the classical factor model to a dynamic one, a lot of various dynamic factor models have been developed and studied from the point of view of parameter estimation. The problem of describing comovements in multivariate time series by means of some nearly independent factors becomes more and more important when facing economic crises and looking for predictions.

In this model the components of a multivariate time series, e.g., financial or economic data observed at regular time intervals, are described by a relatively small number of uncorrelated factors. The usual factor model of multivariate analysis cannot be applied immediately as the factor process also varies in time. Hence, there is a dynamic part, added to the usual linear factor model, the autoregressive process of the factors. The main point of the model is that the components of the underlying multivariate stochastic process are, apart from noise, linear functions of the same dynamic factors that can be identified with some latent driving forces of the whole process. Based on factor loadings, factors can be identified by an expert, and forecasts for the components can be made.

Methods for parameter estimation were also developed. Geweke and Singleton (1981) in [9] gave maximum likelihood estimates of the factors, while Deistler et al. (2005, 2007) used linear algebraic methods, further first-order autoregressive dynamics for the factors and idiosyncratic terms for the errors in [5, 6]. Bánkövi et al. (1981, 1983) in [1, 2] introduced an iteration that uses regression methods and principal components to find the factors one by one; they applied their results for Hungarian macroeconomic data spanning 1953-1979. (Their method is based on the work of Box and Tiao (1977) using canonical transformations of multiple time series in [4].) Here we improve this algorithm so that we are able to extract dynamic factors simultaneously, rather than sequentially. As the input of the algorithm, we have observations for an n-dimensional random vector in equidistant dates between t_1 and t_2. Here n is not necessarily larger than $t_2 - t_1 + 1$, cf. Stock and Watson (2002), [10]. For a given positive integer $k < n$ (k is usually much less than n) we are looking for uncorrelated factors satisfying both a linear and an autoregressive model. The lag length, that is the order of the autoregressive model is the same for the factors and is in the range $[1, 4]$. To estimate the model's parameters we minimize a quadratic cost function on conditions concerning the orthogonality of the factors, the variances of the factors, and the weights balancing between the dynamic and the static part.

At the end, we will extract 3 factors out of 10 yearly observed Hungarian macroeconomic indicators spanning 1993-2007, and try to explain the factor processes based on their loadings; further, we make predictions for 1-2 years ahead.

1 The model

The input data are n-dimensional observations $\mathbf{y}(t) = (y_1(t), \dots, y_n(t))$, where t is the time and the process is observed at equidistant dates between two limits ($t = t_1, \dots, t_2$). For a given positive integer $k < n$ we are looking for (at all leads) uncorrelated factors $f_1(t), \dots, f_k(t)$ such that they satisfy the following model equations.

1. The first one is the linear model

$$f_m(t) = \sum_{i=1}^{n} b_{im} y_i(t), \qquad t = t_1, \dots, t_2; \quad m = 1, \dots, k. \tag{1}$$

2. The second one is the dynamic equation of the factors

$$\hat{f}_m(t) = c_{m0} + \sum_{j=1}^{\ell} c_{mj} f_m(t-j), \qquad t = t_1 + \ell, \dots, t_2; \quad m = 1, \dots, k, \tag{2}$$

   where the lag length ℓ is a given positive integer and $\hat{f}_m(t)$ is the ℓth order auto-regressive prediction of the mth factor at date t (the white-noise term is omitted, therefore we use $\hat{f}_m$ instead of f_m).

3. The third one is the linear prediction of the variables by the factors as in the usual factor model:

$$\hat{y}_i(t) = d_{0i} + \sum_{m=1}^{k} d_{mi} f_m(t), \qquad t = t_1, \dots, t_2; \quad i = 1, \dots, n. \tag{3}$$

   (The idiosyncratic disturbances are also omitted, that is why we use the notation $\hat{y}_i$ instead of y_i.)

We want to estimate the parameters of the model: $\mathbf{B} = (b_{im})$, $\mathbf{C} = (c_{mk})$, $\mathbf{D} = (d_{mi})$ ($m = 1, \dots, \ell$; $i = 1, \dots, n$; $k = 1, \dots \ell$) in matrix notation (estimates of the parameters c_{m0}, d_{0i} can be expressed in terms of these) such that the objective function

$$w_0 \cdot \sum_{m=1}^{\ell} \operatorname{Var}(f_m - \hat{f}_m)_\ell + \sum_{i=1}^{n} w_i \cdot \operatorname{Var}(y_i - \hat{y}_i) \tag{4}$$

is minimum on the conditions for the orthogonality and variance of the factors:

$$\operatorname{cov}(f_m, f_h) = 0, \quad m \neq h; \qquad \operatorname{Var}(f_m) = v_m, \quad m = 1, \dots, k. \tag{5}$$

In (4), the subscript ℓ indicates that the time variation is restricted to dates $t_1 + \ell, \dots, t_2$ only, $w_0, w_1, \dots, w_n$ are given non-negative constants (balancing between the dynamic and static part), while the positive numbers v_m's are the variances of the individual factors indicating their relative importance.

Theoretically, the time series are supposed to be weakly stationary, but in practice, many time series exhibit nonstationary behavior; especially in our example, where each macroeconomic variable might be represented as some aggregate of one or more common inputs. Nonstationarity can be helped by preliminary filtering, whitening, or correction for seasonality, see Deistler and Zinner (2007) [6]. However, we do not use these techniques as it would destroy the so-called adding-up property (see Section 4), except if all the n time series had the same trend and seasonality that is rarely the case. Note that in [4] Box and Tiao (1977) prove that the most predictable components often approach nonstationarity and the least predictable are stationary or independent; they decompose the space of observations into independent, stationary and nonstationary subspaces. We do not subtract the means as well; in fact, means are not intrinsic as we merely use the covariances of the components. Thus, the factors will not have zero means, either.

2 Parameter estimation

First we introduce some notation.

$$\bar{y}_i = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} y_i(t)$$

the sample mean of the ith component, while

$$\mathrm{cov}\,(y_i, y_j) = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} (y_i(t) - \bar{y}_i) \cdot (y_j(t) - \bar{y}_j)$$

stands for the sample covariance and

$$\mathrm{cov}^*(y_i, y_j) = \frac{1}{t_2 - t_1} \sum_{t=t_1}^{t_2} (y_i(t) - \bar{y}_i) \cdot (y_j(t) - \bar{y}_j)$$

for the corrected empirical covariance between the i-th and j-th components. By the notation

$$Y_{ij} := \mathrm{cov}\,(y_i, y_j), \qquad i, j = 1, \dots n,$$

let $\mathbf{Y} := (Y_{ij})$ be the $n \times n$ symmetric, positive semidefinite sample covariance matrix (sometimes we use the corrected one).

Observe, that the parameters c_{m0}, d_{0i} can be written in terms of the other parameters:

$$c_{m0} = \frac{1}{t_2 - t_1 - \ell + 1} \sum_{t=t_1+\ell}^{t_2} (f_m(t) - \sum_{j=1}^{\ell} c_{mj} f_m(t-j)), \qquad m = 1, \dots, k$$

and

$$d_{0i} = \bar{y}_i - \sum_{m=1}^{k} d_{mi} \bar{f}_m, \qquad i = 1, \dots, n.$$

Thus, the parameters to be really estimated are entries of the $n \times k$ matrix $\mathbf{B}$, the $k \times n$ matrix $\mathbf{D}$, and the $k \times \ell$ matrix $\mathbf{C}$. Let us denote by $\mathbf{b}_m \in \mathbb{R}^n$ the m-th column of the matrix $\mathbf{B}$.

We also define the lagged time series

$$z_i^m(t) = y_i(t) - \sum_{j=1}^{\ell} c_{mj} y_i(t-j), \quad t = t_1 + \ell, \dots, t_2;\ i = 1, \dots, n;\ m = 1, \dots, k \tag{6}$$

and its empirical covariance matrix of entries

$$Z_{ij}^m := \mathrm{cov}\,(z_i^m, z_j^m) = \frac{1}{t_2 - t_1 - \ell + 1} \sum_{t=t_1+\ell}^{t_2} (z_i^m(t) - \bar{z}_i^m) \cdot (z_j^m(t) - \bar{z}_j^m), \tag{7}$$

where $\bar{z}_i^m = \frac{1}{t_2 - t_1 - \ell + 1} \sum_{t=t_1+\ell}^{t_2} z_i^m(t)$, $i = 1, \dots, n$; $m = 1, \dots, k$. Further, let $\mathbf{Z}^m = (Z_{ij}^m)$ be the $n \times n$ symmetric, positive semidefinite covariance matrix of the lagged variables, $m = 1, \dots, k$.

To write the objective function (4) in terms of these quantities, we make the following argument:

$$f_m(t) - \hat{f}_m(t) = \sum_{j=1}^{n} b_{jm} z_j^m(t) - c_{m0}$$

and

$$\mathrm{Var}\,(F_m - \hat{F}_m)_\ell = \mathbf{b}_m^T \mathbf{Z}^m \mathbf{b}_m. \tag{8}$$

In view of (1),

$$\mathrm{Var}\,(f_m) = \mathbf{b}_m^T \mathbf{Y} \mathbf{b}_m, \qquad m = 1, \dots, k$$

and

$$\mathrm{cov}\,(y_i, f_m) = \sum_{j=1}^{n} b_{jm} Y_{ij}, \quad i = 1, \dots, n; \quad m = 1, \dots, k.$$

Further, due to the orthogonality of the factors, and due to the equation (3)

$$\begin{aligned}
\mathrm{Var}\,(y_i - \hat{y}_i) &= Y_{ii} - 2 \sum_{m=1}^{k} d_{mi} \mathrm{cov}\,(y_i, f_m) + \sum_{m=1}^{k} d_{mi}^2 v_m \\
&= Y_{ii} - 2 \sum_{m=1}^{k} d_{mi} \sum_{j=1}^{n} b_{jm} Y_{ij} + \sum_{m=1}^{k} d_{mi}^2 v_m.
\end{aligned}$$

With these, the objective function (4) to be minimized is

$$\begin{aligned}
G(\mathbf{B}, \mathbf{C}, \mathbf{D}) = w_0 \sum_{m=1}^{k} \mathbf{b}_m^T \mathbf{Z}^m \mathbf{b}_m + \sum_{i=1}^{n} w_i Y_{ii} - 2 \sum_{i=1}^{n} w_i \sum_{m=1}^{k} d_{mi} \sum_{j=1}^{n} b_{jm} Y_{ij} \\
+ \sum_{i=1}^{n} w_i \sum_{m=1}^{M} d_{mi}^2 v_m,
\end{aligned}$$

where the minimum is taken on the constraints

$$\mathbf{b}_m^T \mathbf{Y} \mathbf{b}_h = \delta_{mh} \cdot v_m, \qquad m, h = 1, \dots, k. \tag{9}$$

The procedure finding the minimum is based on the following iteration that consists of an outer and an inner cycle. Choosing an initial $\mathbf{B}^{(0)}$ of columns satisfying (9), the following three steps are alternated in the tth outer iteration.

1. Starting with $\mathbf{B}^{(t)}$ we calculate the f_m's based on (1), then we fit a linear model to estimate the parameters of the autoregressive model (2). Hence, the current value $\mathbf{C}^{(t)}$ is obtained.

2. Based on this $\mathbf{C}^{(t)}$, we find matrices $\mathbf{Z}^m$ using (6) and (7) (actually, to obtain $\mathbf{Z}^m$, the m-th row of $\mathbf{C}$ is needed only), $m = 1, \dots, k$. This $\mathbf{Z}^m$ also depends on t, however, to simplify notation, we do not indicate this dependence. Putting this auxiliary variable into $G(\mathbf{B}^{(t)}, \mathbf{C}^{(t)}, \mathbf{D})$, we take its minimum with respect to $\mathbf{D}$, while keeping $\mathbf{B}$ and $\mathbf{C}$ fixed. The minimum is taken at $\mathbf{D}^{(t)}$.

3. Now keeping $\mathbf{C}$ and $\mathbf{D}$ fixed, we minimize $G(\mathbf{B}, \mathbf{C}^{(t)}, \mathbf{D}^{(t)})$ with respect to $\mathbf{B}$. This minimization needs an inner cycle. The minimum is taken at $\mathbf{B}^{(t+1)}$.

With this new $\mathbf{B}$ we return to Step 1 of the outer cycle ($t := t+1$) and proceed until convergence. As the value of the nonnegative objective function is in each step decreased we might expect its value stabilized, but only the convergence to a local minimum can be guaranteed.

The inner cycle is described in the next section, here we discuss Step 2 and preparation of Step 3 in details.

Step 2: Fixing $\mathbf{C}$, the part of the objective function to be minimized in $\mathbf{B}$ and $\mathbf{D}$ is

$$g(\mathbf{B}, \mathbf{D}) = w_0 \sum_{m=1}^{k} \mathbf{b}_m^T \mathbf{Z}^m \mathbf{b}_m + \sum_{i=1}^{n} w_i \sum_{m=1}^{k} d_{mi}^2 v_m - 2 \sum_{i=1}^{n} w_i \sum_{m=1}^{k} d_{mi} \sum_{j=1}^{n} b_{jm} Y_{ij}$$

that is first optimized in $\mathbf{D}$. To this end, we solve the equation

$$\frac{\partial f(\mathbf{B}, \mathbf{D})}{\partial d_{mi}} = 2 w_i v_m d_{mi} - 2 w_i \sum_{j=1}^{n} b_{mj} Y_{ij} = 0$$

separately for the entries of $\mathbf{D}$. It is easy to see that the matrix $\mathbf{D}^{opt}$ of entries

$$d_{mi}^{opt} = \frac{1}{v_m} \sum_{j=1}^{n} b_{jm} Y_{ij}$$

gives a local minimum of $g(\mathbf{B}, \mathbf{D})$ for fixed $\mathbf{B}$.

Step 3: Putting back the so obtained $\mathbf{D}^{opt}$ into $g(\mathbf{B}, \mathbf{D})$, it will have the following form:

$$g(\mathbf{B}, \mathbf{D}^{opt}) = w_0 \sum_{m=1}^{\ell} \mathbf{b}_m^T \mathbf{Z}^m \mathbf{b}_m - \sum_{i=1}^{n} w_i \sum_{m=1}^{\ell} \frac{1}{v_m} (\sum_{j=1}^{n} b_{jm} Y_{ij})^2.$$

From this, by introducing the $n \times n$ symmetric matrix $\mathbf{V} = (V_{jh})$ of entries $V_{jh} = \sum_{i=1}^{n} w_i Y_{ij} Y_{ih}$ and the $n \times n$ symmetric matrix

$$\mathbf{S}_m = w_0 \mathbf{Z}^m - \frac{1}{v_m} \mathbf{V}, \qquad m = 1, \dots, k,$$

we have

$$g(\mathbf{B}, \mathbf{D}^{opt}) = \sum_{m=1}^{k} \mathbf{b}_m^T \mathbf{S}_m \mathbf{b}_m \tag{10}$$

that is to be minimized on the constraints for $\mathbf{b}_m$'s.

To apply the algorithm to be introduced in Section 3, we have to transform the vectors $\mathbf{b}_1, \dots, \mathbf{b}_m$ into an orthonormal set. Because of the constraints, the transformations

$$\mathbf{x}_m := \frac{1}{\sqrt{v_m}} \mathbf{Y}^{1/2} \mathbf{b}_m, \quad \mathbf{A}_m := v_m \mathbf{Y}^{-1/2} \mathbf{S}_m \mathbf{Y}^{-1/2}, \quad m = 1, \dots, k \tag{11}$$

will result in an orthonormal set $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$; further

$$\mathbf{b}_m^T \mathbf{S}_m \mathbf{b}_m = \mathbf{x}_m^T \mathbf{A}_m \mathbf{x}_m, \qquad m = 1, \dots, k,$$

and hence,

$$F(\mathbf{B}, \mathbf{D}^{opt}) = \sum_{m=1}^{k} \mathbf{x}_m^T \mathbf{A}_m \mathbf{x}_m. \tag{12}$$

The sum of the heterogeneous quadratic forms of (12) is minimized by the algorithm of the next section (inner cycle). Let $\mathbf{x}_1^{opt}, \dots, \mathbf{x}_k^{opt}$ denote the orthonormal set giving the minimum. Inverting the first transformation of (11), the vectors

$$\mathbf{b}_m^{opt} = \sqrt{v_m} \mathbf{Y}^{-1/2} \mathbf{x}_m^{opt}, \qquad m = 1, \dots, k$$

will give the column vectors of $\mathbf{B}^{opt}$, minimizing $F(\mathbf{B}, \mathbf{D}^{opt})$.

3 Compromise systems of symmetric matrices

Given the $n \times n$ symmetric matrices $\mathbf{A}_1, \dots, \mathbf{A}_k$ ($k \leq n$) we are looking for an orthonormal set of vectors $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ for which

$$\sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A}_i \mathbf{x}_i$$

is maximum.

The theoretical solution is obtained by Lagrange's multipliers: the $\mathbf{x}_i$'s giving the optimum satisfy the system of linear equations

$$A(\mathbf{X}) = \mathbf{X}\mathbf{S} \tag{13}$$

with some $k \times k$ symmetric matrix $\mathbf{S}$ (its entries are the multipliers), where the $n \times k$ matrices $\mathbf{X}$ and $A(\mathbf{X})$ consist of the following columns:

$$\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_k), \quad A(\mathbf{X}) = (\mathbf{A}_1\mathbf{x}_1, \dots, \mathbf{A}_k\mathbf{x}_k).$$

Due to the constraints imposed on $\mathbf{x}_1, \dots, \mathbf{x}_k$, the non-linear system of equations

$$\mathbf{X}^T \mathbf{X} = \mathbf{I}_k \tag{14}$$

must also hold. As $\mathbf{X}$ and the symmetric matrix $\mathbf{S}$ contain altogether $nk + k(k+1)/2$ free parameters, while the conditions (13) and (14) contain the same number of equations, a solution of the problem is expected. Transforming (13) into a homogeneous system of linear equations, a non-trivial solution of it exists, if

$$|\mathbf{A} - \mathbf{I}_n \otimes \mathbf{S}| = 0, \tag{15}$$

where the $nk \times nk$ matrix $\mathbf{A}$ is a Kronecker-sum $\mathbf{A} = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_k$ and $\otimes$ denotes the Kronecker-product.

Equation (15) is reminiscent of the characteristic equation, being a polynomial of degree $k(k+1)/2$ of the in- and upper-diagonal entries of the "compromise matrix" $\mathbf{S}$. The exact solution is not known, numerical methods are to be applied. Instead, in [3] – Bolla et al. (1998) – an iteration was introduced.

Starting with a suborthogonal matrix $\mathbf{X}^{(0)}$ (of orthonormal columns), the mth step of the iteration based on the $(m-1)$th one is as follows ($m = 1, 2, \dots$). Take the polar decomposition of $A(\mathbf{X}^{(m-1)})$ into an $n \times k$ suborthogonal matrix $\mathbf{X}^{(m)}$ and a $k \times k$ symmetric matrix $\mathbf{S}^{(m)}$. Let the first factor be the next $\mathbf{X}^{(m)}$, and continue until convergence. The polar decomposition is obtained by SVD. In [3] the convergence of the algorithm was also proved. We remark that the trace of the second factor $\mathbf{S}^{(m)}$ converges to the optimum of the objective function.

The above iteration is easily adopted to negative semidefinite or indefinite matrices and to minima instead of maxima in the following way. Find the minimum of

$$\sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A}_i \mathbf{x}_i$$

on the constraints (14), where $\mathbf{A}_1, \dots, \mathbf{A}_n$ are $n \times n$ symmetric matrices. Let λ_i^{max} denote the largest eigenvalue of $\mathbf{A}_i$ ($i = 1, \dots, k$), and set

$$\lambda := \max_{i \in \{1,\dots,k\}} \lambda_i^{max} + \varepsilon,$$

where ε is an arbitrarily small positive constant. The matrices

$$\tilde{\mathbf{A}}_i := -\mathbf{A}_i + \lambda \mathbf{I}_n, \qquad i = 1, \dots, k$$

are positive definite and

$$\min \sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A}_i \mathbf{x}_i = -\max \sum_{i=1}^{k} \mathbf{x}_i^T (-\mathbf{A}_i) \mathbf{x}_i = -\max \sum_{i=1}^{k} \mathbf{x}_i^T \tilde{\mathbf{A}}_i \mathbf{x}_i + \lambda k,$$

further, the minimum of the first sum is taken on the same $\mathbf{x}_i$'s as the maximum of the last one in terms of $\tilde{\mathbf{A}}_i$'s.

4 Application to macroeconomic data

We used aggregate data of the Hungarian Statistical Office. We consider 10 highly correlated macroeconomic time series of the Hungarian Republic, registered yearly, spanning 1993–2007. Note that in macroeconomic forecasting, the number of predictor series (n) can be very large, often larger than the time series observations ($t_2 - t_1 + 1$), see Stock and Watson (2002).

Names and mnemonics of the components are as follows.

- Gross Domestic Product (1000 million HUF) – GDP
- Number of Students in Higher Education – EDU
- Number of Hospital Beds – HEALTH
- Industrial Production (1000 million HUF) – IND
- Agricultural Area (1000 ha) – AGR
- Energy Production (petajoule) – ENERGY

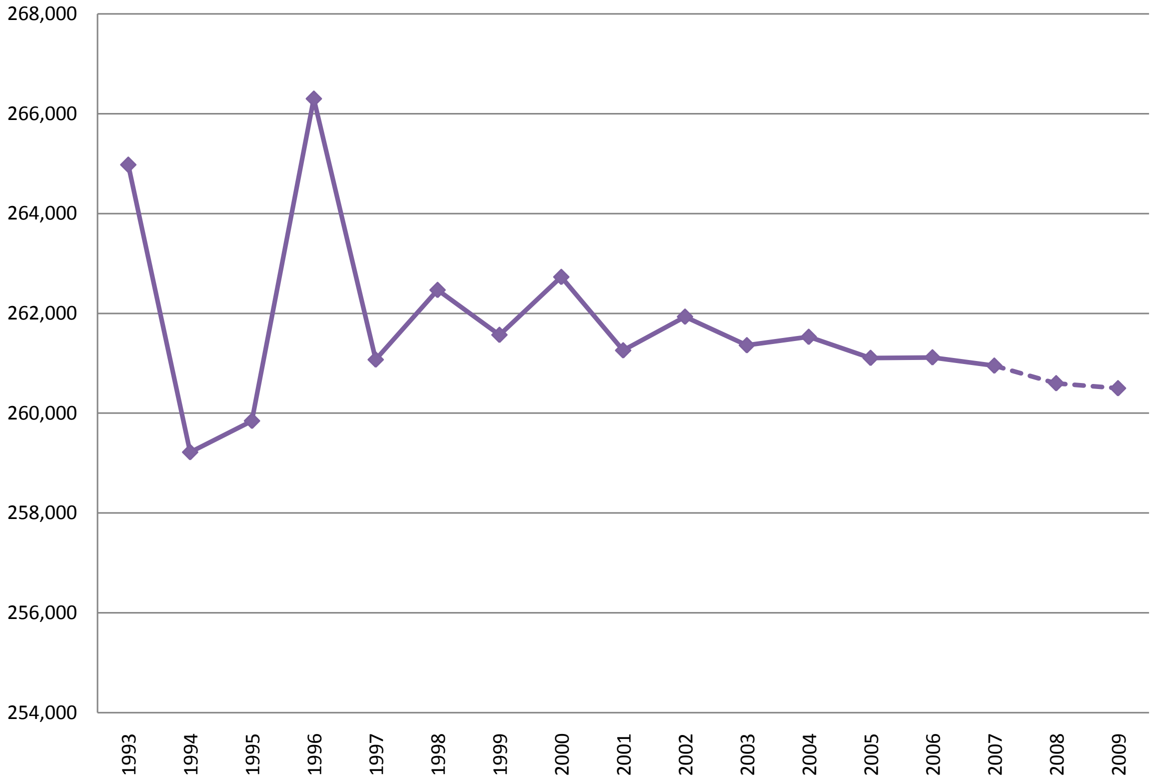

Figure 1: Dynamic factor 1

- Energy Import (petajoule) – IMP
- Energy Export (petajoule) – EXP
- National Economic Investments (1000 million HUF) – INV
- Number of Publications – INNOV

We extracted 3 factors out of the data, using lag length 4. As the variables were measured in different units we normalized them such that we made adjustments, where necessary so as to produce numbers of comparable magnitude in the different series; later we used the reciprocals of their standard deviations as weights $w_1, \ldots, w_n$ in the objective function (4). In [1], authors use the same weights $v_m = t_2 - t_1 + 1$ $(m = 1, \ldots, k)$ for the factors, we also used these weights; further, we used the suggested choice $w_0 = n/v_m$ ensuring the equilibrium between the dynamic and static parts.

In Figure 1, the first factor demonstrates a decrease, then an increase, and reaches its peak in 1996 (when restrictions on government spending and social benefits were introduced and investments started). Since 1997 this factor has

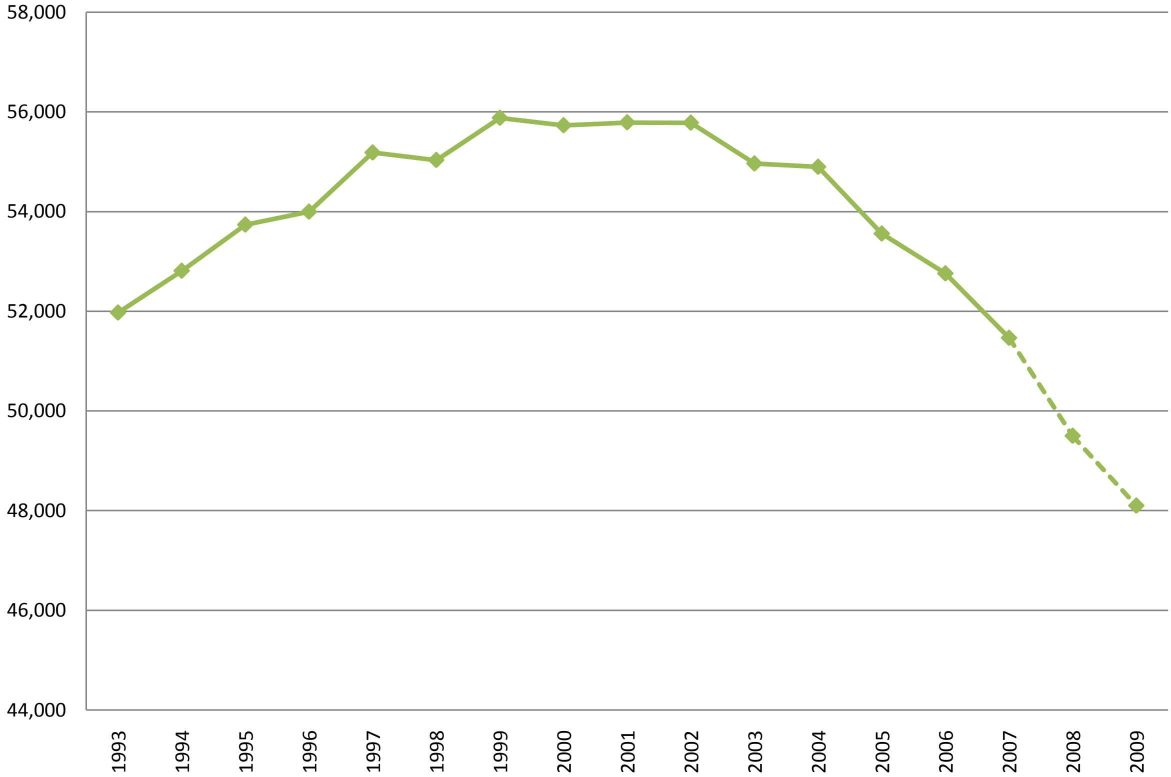

Figure 2: Dynamic factor 2

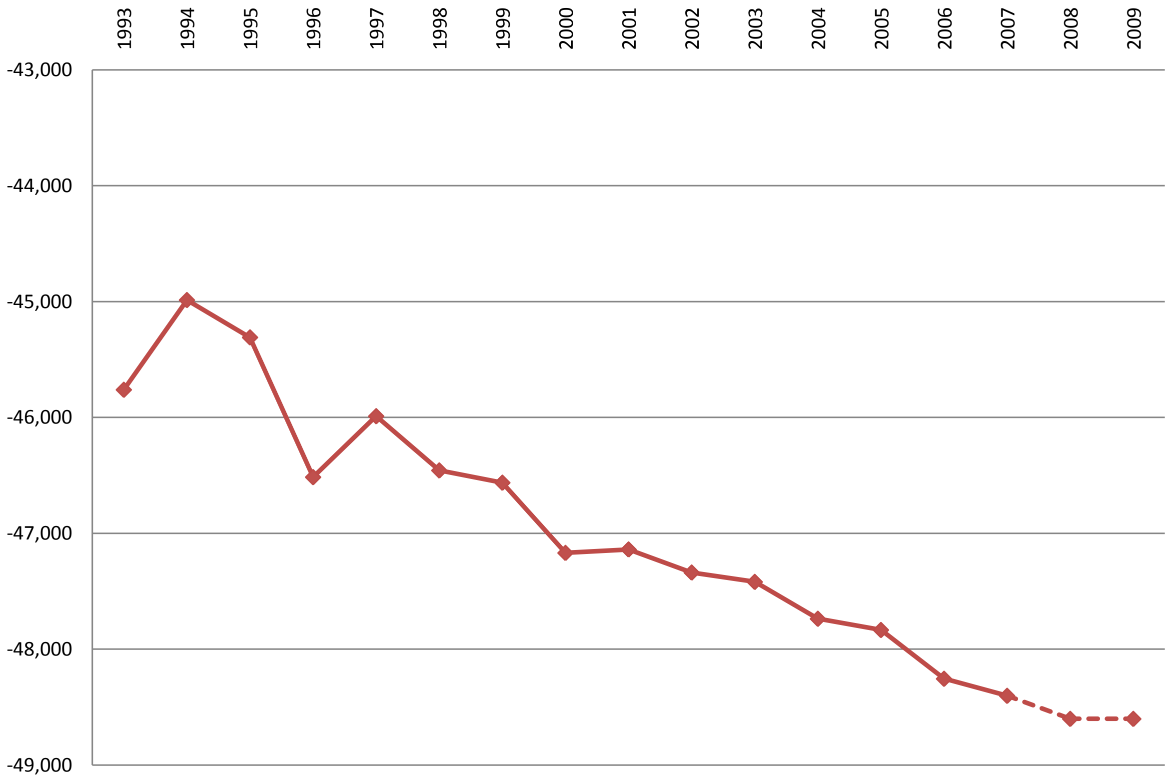

Figure 3: Dynamic factor 3

made slight periodic movements. Based on Table 1, variables GDP, ENERGY, and HEALTH are mainly responsible for this factor (in the middle of the 1990's there were also reforms in the health care system). In Figure 2, the second factor slowly increases, then decreases, with highest values around the turn of the century. The variables EDU, ENERGY, and AGR have the highest coefficients in it. Note that the number of students in higher education steadily increased in the 1990's, however, since the beginning of the century the interest in some areas of study has dropped as people with higher degrees had difficulties finding jobs. As Figure 3 demonstrates, the third factor is somewhat antipodal to the first one, with highest absolute value coefficients in GDP, ENERGY, and HEALTH; further, it shows smaller fluctuations. Future analysis is required to obtain a reasonable explanation for this phenomenon. Possibly, the first two factors are only significant, while the next ones are damped imitations of them. We remark that in our model k is, in fact, the maximum number of factors, which does not contradict to certain rank conditions, see e.g., Deistler and coauthors (2005, 2007). The actual number of factors can be less, depending on the least square errors and practical considerations; it is the expert who decides how many factors to retain.

	factor 1	**factor 2**	**factor 3**
GDP	38.324	-2.541	-6.116
EDU	-1.775	5.725	0.015
HEALTH	10.166	0.837	-1.650
IND	-0.261	0.255	-0.107
AGR	6.146	2.919	-1.124
ENERGY	24.082	4.592	-4.054
IMP	1.560	-1.209	-0.213
EXP	-3.907	-0.233	0.615
INV	2.864	0.038	-0.510
INNOV	-0.608	0.197	0.089

Table 1: Factor Expressed in Terms of the Variables (matrix **B**)

	factor 1	**factor 2**	**factor 3**	**Constant term**
GDP	-0.108	-0.025	-0.677	-0.670
EDU	-0.142	0.145	-0.877	-8.637
HEALTH	0.115	-0.132	0.656	16.250
IND	-0.898	-0.187	-5.784	-14.690
AGR	0.021	0.005	0.137	6.809
ENERGY	0.085	-0.038	0.543	10.055
IMP	-0.098	-0.152	-0.868	0.311
EXP	-0.516	-0.931	-1.840	109.915
INV	-0.209	0.026	-1.341	-6.779
INNOV	-0.061	0.121	-0.484	-9.867

Table 2: Variables Estimated By the Factors (matrix **D**)

The coefficients of matrices **B**, **D**, and **C** are shown in Tables 1, 2, and 3, respectively. The relatively high constant terms in the linear prediction of the variables by the factors (see Table 2) refer to "small" communalities. However, the constant coefficients in the autoregressive model are small (see Table 3) and the coefficient belonging to lag 2 is the largest. Notice that since 1990, different governments changed each other in every 4 years, and lag 2 corresponds to the mid-period, when the measures introduced by the new government probably had the higher impact on the economy.

We also made predictions for the factors for 2 years ahead by means of matrix **C**. The predicted factor values for 2008 and 2009 are denoted by dashed lines and show decline in all the three factors, possibly indicating the evolving economic crisis. Based on matrix **D**, we predicted the variables by the factors for the period 1993-2007 and calculated the static part of the objective function, which represents one possible source of the error in the algorithm. We also made predictions for 2008 and 2009 based on the predicted values of the factors. Data for 2009 are not available yet, however, the 2008's estimates showed a good fit to the factual data in case of most variables. We found that the squared error 1.16 of this only year is comparable to the cumulated error 11.54 of 15 years.

Bánkövi et. al. (1981) in [1] prove that the model equations (1)–(3) are linearly syntonic, and hence, the adding-up constraints of Denton (1978) (see [7]) are satisfied. This justifies the correctness of the above way of forecasting via the factors.

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Time lag	**factor 1**	**factor 2**	**factor 3**
0	-0.000	0.001	-0.000
1	0.069	0.283	0.117
2	0.473	1.644	0.495
3	0.205	0.229	0.141
4	0.251	-1.168	0.258

Table 3: Dynamic Equations of the Factors (matrix **C**)

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