Exercise sheet 1, Analysis 2, 2023

Exercises marked with E are meant to be fairly easy, only requiring the knowledge of basic definitions. Exercises marked with $*$ are meant to be hard in some sense: either the solution requires a clever trick, or the solution is long and cumbersome. Unmarked exercises are meant to be of medium difficulty.

(1) ^{E} Prove that any finite set $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ is Jordan measurable and $\lambda_J(A) = 0$.

(2) Let $A = \mathbb{Q} \cap [0,1]$. Prove that $\lambda_J^*(A) = 1$ but $\lambda_{*,J}(A) = 0$, therefore A is not Jordan measurable. (This means that the historically used phrase "Jordan measure" is misleading because it is *not a measure*, as the Jordan measurable sets do not form a σ-algebra.)

(3) Show that in the definition of Jordan inner measure we could equivalently take countable unions (instead of finite unions), i.e. $\sup\{\sum_{n=1}^{\infty} \lambda(T_n) : \dot{\cup} T_n \subset A\} = \lambda_{*,J}(A)$.

(4) ^{E} Let $X = \{x_1, x_2, x_3\}$ be a set with three elements. Construct all possible σ-algebras over X.

(5) ^{E} Prove that any σ-algebra $\mathcal{A}$ is closed under countable intersection, i.e. if $A_n \in \mathcal{A}$ then $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$.

(6) ^{E} Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E, F \in \mathcal{M}$. Prove that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

(7) ^{E} Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E \in \mathcal{M}$. Let $\mu_E(A) = \mu(A \cap E)$ for any $A \in \mathcal{M}$. Prove that μ_E is also a measure on $(X, \mathcal{M})$.

Exercise sheet 2, Analysis 2, 2023

(8) ^{E} Let X be any non-empty set, and $A \subset X$. Determine the σ-algebra generated by the set systems
(a) $\{A\}$
(b) $\{B : B \subset A\}$

(9) Consider the open sets in a metric space (X, d). Do they form a σ-algebra? Determine the special case when they do.

(10) ^{E} Let $f : X \to Y$ be any function. Prove that
(a) if $\mathcal{B}$ is a σ-algebra on Y, then $\{f^{-1}(B) : B \in \mathcal{B}\}$ is also a σ-algebra on X. (This is called the "pull-back σ-algebra.)
(b) If $\mathcal{A}$ is a σ-algebra on X, then $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is also a σ-algebra on Y. (This is called the "push-forward" σ-algebra.)

(11) Prove that the intervals $\{(a, b] : a \leq b\}$ generate the Borel σ-algebra on $\mathbb{R}$. (Recall, that the Borel σ-algebra, by definition, is generated by the open sets in $\mathbb{R}$.)

(12) * Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra on $\mathbb{R}$, and let $Y \subset \mathbb{R}$ be any set. Prove that the Borel σ-algebra on Y is given by $\mathcal{B}(Y) = \{A \cap Y : A \in \mathcal{B}(\mathbb{R})\}$.

(13) Let $A \subset \mathbb{R}^d$. In the definition of the Jordan outer measure we used a finite number of boxes to cover A. Instead, we define the *Lebesgue outer measure* by allowing countably many boxes:
$\lambda^*(A) = \inf\{\sum_{i=1}^{\infty} \lambda(T_i) : A \subset \cup_{i=1}^{\infty} T_i\}$,
where $\lambda(T_i)$ is the product of the sides of T_i. Prove that for $A = \mathbb{Q} \cap [0, 1]$ we have $\lambda^*(A) = 0$ (and hence we see that the Lebesgue outer measure is not the same as the Jordan outer measure!)

(14) ^{E} Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$, and $\overline{\mathcal{A}} = \{E \cup F : E \in \mathcal{A}, F \subset N \text{ for some } N \in \mathcal{N}\}$. Prove that $\overline{\mathcal{A}}$ is also a σ-algebra on X.

(15) In the previous exercise, let $\overline{\mu}(E \cup F) = \mu(E)$. Prove that $\overline{\mu}$ is a measure on $\overline{\mathcal{A}}$. Prove that it has the following completeness property: if $\overline{\mu}(A) = 0$ for a set $A \in \overline{\mathcal{A}}$, then every subset $B \subset A$ is also measurable and $\overline{\mu}(B) = 0$. (The space $(X, \overline{\mathcal{A}}, \overline{\mu})$ is called the *completion* of $(X, \mathcal{A}, \mu)$.)

Exercise sheet 3, Analysis 2, 2023

In the exercises below, λ always denotes the Lebesgue measure.

(16) ^{E} Let $A \subset \mathbb{R}^d$ be a set such that its Lebesgue outer measure is 0. Prove A is Lebesgue measurable and $\lambda(A) = 0$. (Hint: use the definition of measurability: the splitting property.)

(17) ^{E} Let $X = [0,1]^2$, and let $\mathcal{E}$ denote the family of rectangles $T_{ab} = \{(x,y) \in X : 0 \le a \le x \le b \le 1, 0 \le y \le 1\}$. Let $\rho : \mathcal{E} \to [0,1]$, $\rho(T_{ab}) = b - a$. Let μ^* be the outer measure generated by ρ. Prove that the diagonal $D = \{(x,y) \in X : x = y\}$ is not measurable (i.e. it does not have the splitting property).

(18) ^{E} Prove that if $E \subset [0,1]$, $\lambda(E) = 1$, then E is dense in $[0,1]$. (Hint: prove equivalently that if a set is not dense, then its complement must contain an open interval, and therefore its measure must be smaller than one.)

(19) Prove that the Lebesgue measure on $\mathbb{R}^d$ is *open regular*, i.e. for any measurable set E we have $\lambda(E) = \inf\{\lambda(U) : U \supset E, U \text{ is open}\}$.

(20) Prove that the Lebesgue measure on $\mathbb{R}^d$ is *compact regular*, i.e. for any measurable set E we have $\lambda(E) = \sup\{\lambda(K) : K \subset E, K \text{ is compact}\}$.

(21) ^{E} Let $\mathcal{A}_0$ be the family of subsets of $\mathbb{R}^d$ which can be written as finite union of disjoint boxes $\prod_{j=1}^n (a_i, b_i]$. Let $\mathcal{A}_1$ be the family of subsets of $\mathbb{R}^d$ which can be written as finite union of (not necessarily disjoint) boxes $\prod_{j=1}^n (a_i, b_i]$. Prove that $\mathcal{A}_0 = \mathcal{A}_1$, and it is an algebra, but not a σ-algebra. (An *algebra* is a system of sets closed under complements and finite unions. Hint: make a drawing in 2-dimensions.)

(22) In the Caratheodory theorem, let us accept the fact the family of μ^* measurable sets form a σ-algebra, and μ^* is a measure on this σ-algebra. Prove that this measure is complete, as the theorem states.

(23) * Let $H \subset \mathbb{R}$, $\lambda(H) = 0$. Prove that there exists a $c \in \mathbb{R}$ such that all the numbers $c + h$ ($h \in H$) are irrational.

(24) Prove that the Cantor set is compact, nowhere dense, its Lebesgue measure is 0, and it cardinality is continuum.

(25) Prove the Borel-Cantelli lemma: "if A_i are events such that the sum of their probability is finite, then the probability that infinitely many of them occurs is 0". More formally, if μ is a probability measure on a σ-algebra $\mathcal{A}$, and $A_i \in \mathcal{A}$ are such that $\sum_{i=1}^{\infty} \mu(A_i) < +\infty$ then $\mu(\cap_{n \in \mathbb{N}} \cup_{k \ge n} A_k) = 0$. (Note here that the event $\cap_{n \in \mathbb{N}} \cup_{k \ge n} A_k$ describes exactly that infinitely many of the A_i's occur. Hint: use the continuity property of the measure.)

Exercise sheet 4, Analysis 2, 2023

(26) ^{E} Let $(X_i, \mathcal{A}_i)$ be measurable spaces for all positive integers i. Prove that the product σ-algebra $\bigotimes_{i=1}^{\infty} \mathcal{A}_i$ is generated by "generalized boxes" of the form $\prod_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}_i$. (Recall from class, that the product σ-algebra is, by definition, generated by cylinder sets.)

(27) Prove that $\mathcal{B}(\mathbb{R}^d) = \bigotimes_{i=1}^{d} \mathcal{B}(\mathbb{R})$. (Hint: the LHS is generated by open sets, the RHS is generated by generalized boxes. Prove that all open sets can be built up from such boxes and, conversely, all cylinder sets can be built up from open sets.)

(28) * Let $(X, \mathcal{A})$, $(Y, \mathcal{M})$ be measurable spaces, and let $T \in \mathcal{A} \otimes \mathcal{M}$ be a set in the product σ-algebra. Prove that all cross-sections $T_x = \{y \in Y : (x, y) \in T\}$ and $T_y = \{x \in X : (x, y) \in T\}$ are measurable for all $x \in X$ and $y \in Y$.

(29) ^{E} Prove that the Lebesgue-measure is translation invariant, i.e. for all Lebesgue-measurable set $A \subset \mathbb{R}$, and all $t \in \mathbb{R}$, we have $\lambda(A) = \lambda(t + A)$. (Hint: check that the definition of the Lebesgue outer measure is translation invariant.)

(30) ^{E} Let $A = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \leq 1, y > 0\}$. Prove that A is Lebesgue-measurable. (Hint: prove that it is a Borel set, and use the property that all Borel sets are Lebesgue-measurable.)

Exercise sheet 5, Analysis 2, 2023

(31) ^{E} Let μ be a finite Borel measure on $\mathbb{R}$ (i.e. a finite measure on the Borel subsets of $\mathbb{R}$). Let $F(x) = \mu((-\infty, x])$. Prove that F is monotonically increasing and right-continuous. (F is called the distribution function of μ.)

(32) ^{E} Let $F(x) = 0$ if $x < 0$, and $F(x) = 1$ if $x \geq 0$. Follow the construction of the Lebesgue-Stieltjes measure induced by F, and prove that $\mu_F = \delta_0$, the Dirac measure situated at 0.

(33) ^{E} Let $F_1(x) = [x]$ (the integer part of x), and $F_2(x) = 2x$. What are the Lebesgue-Stieltjes measures induced by F_1 and F_2?

(34) ^{E} Let $g : (X, \mathcal{A}) \to (Y, \mathcal{M})$ and $f : (Y, \mathcal{M}) \to (Z, \mathcal{E})$ be measurable functions. Prove that $f \circ g$ is also measurable.

(35) Let $(X, \mathcal{A}), (Y_1, \mathcal{M}_1), (Y_2, \mathcal{M}_2)$ be measurable spaces. Prove that a function $f = (f_1, f_2) : X \to Y_1 \times Y_2$ is $\mathcal{A} - \mathcal{M}_1 \otimes \mathcal{M}_2$-measurable iff f_1 is $\mathcal{A} - \mathcal{M}_1$-measurable, and f_2 is $\mathcal{A} - \mathcal{M}_2$-measurable. (Use the definition of the product σ-algebra.)

(36) ^{E} Let $(X, \mathcal{A}, \mu)$ be a complete measure space, and let $f, g : X \to \mathbb{R}$ be functions such that $f = g$ μ-almost everywhere. Prove that if f is measurable, than so is g.

(37) Let $f : \mathbb{R} \to \mathbb{R}^n$ be a Borel measurable function, and let $g(x) = \|f(x)\|_\infty$. Prove that $g : \mathbb{R} \to \mathbb{R}$ is also Borel measurable.

(38) Let $f, g : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions. Prove that $f + g$ is also Borel measurable.

(39) Let $f \in L^+(X)$. Prove that $\int_X f d\mu = 0$ iff $f(x) = 0$ for μ-almost every $x \in X$. (Hint: use the definition of $\int_X f d\mu$.)

(40) ^{E} Let $X = \mathbb{N}$, and μ the counting measure (i.e. $\mu(A) = |A|$ if A is a finite set, and $\mu(A) = \infty$ if A is not finite). Prove that in this case $L^+(X)$ is the set of nonnegative sequences, and for any $f = (a_n) \in L^+$ we have $\int_X f d\mu = \sum_{n=0}^{\infty} a_n$. (Hint: use the definition of $L^+(X)$, and the definition of $\int_X f d\mu$.)

(41) ^{E} Let $(X, \mathcal{A})$ be a measurable space, and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \ldots$ be measures on it. Prove that $\mu(E) = \sup_{n \in \mathbb{N}} \mu_n(E)$ is also a measure on $(X, \mathcal{A})$-n. (Hint: use the previous exercise and the monotone convergence theorem.)

(42) ^{E} Give an example of a sequence of functions $f_n \in L^+(\mathbb{R})$, $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$, but $\int f \neq \lim_{n\to\infty} \int f_n$.

(43) ^{E} Prove that $\int_0^1 x^a (1-x)^{-1} \log x \; dx = -\sum_{k=1}^{\infty} (a+k)^{-2}$ for all $a > -1$. (Hint: take the power series of $(1-x)^{-1}$ and use the monotone convergence theorem.)

(44) ^{E} Calculate $\lim_{n\to\infty} \int_0^n e^{-2x}(1+\frac{x}{n})^n dx$. (Hint: use the monotone convergence theorem or the Lebesgue dominated convergence theorem.)

(45) ^{E} Calculate $\lim_{n\to\infty} \int_1^n \frac{\pi + 2x^2 \arctan(nx)}{2x^4 + \sin\frac{1}{n}} dx$. (Hint: use the Lebesgue dominated convergence theorem.)

(46) ^{E} Calculate $\int_0^\infty \frac{n \sin(x/n)}{x(x^2+1)} dx$. (Hint: for $t = x/n$ we have $\frac{\sin t}{t} \to 1$, and then use the Lebesgue dominated convergence theorem.)

(47) ^{E} Calculate $\lim_{n\to\infty} \int_0^1 \frac{nx^n}{1+x} dx$. (Hint: first integrate by parts, and then use Lebesgue dominated convergence theorem.)

(48) Let $F(x) = \arctan(2x+1)$ if $x < 0$ and $F(x) = \frac{2x+1}{x+1}$ if $x \geq 0$. Let μ be the Lebesgue-Stieltjes measure induced by F, i.e. $\mu((a,b]) = F(b) - F(a)$. Calculate $\int_{-1}^1 (2+x) d\mu$. (Hint: do not forget the jump of F at 0.)

Exercise sheet 7 (summary before midterm 1), Analysis 2, 2023

(49) ^{E} Let $A_1 = [0,1)$, $A_2 = (2,4)$. Determine the σ-algebra on $\mathbb{R}$ generated by A_1, A_2.

(50) ^{E} Let $(X, \mathcal{A})$, $(Y, \mathcal{M})$ be measurable spaces. Let $\mathcal{N} = \{E \subset X \cap Y : E \in \mathcal{A} \cap \mathcal{M}\}$. Is it true that $\mathcal{N}$ is a σ-algebra on $X \cap Y$?

(51) ^{E} Let $A = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1, x > 0\}$. Prove that A is Lebesgue measurable, and calculate its measure $\lambda(A)$.

(52) ^{E} Prove that for any non-empty open set $U \subset \mathbb{R}^n$ we have $\lambda(U) \neq 0$.

(53) ^{E} Let H be a Lebesgue measurable set in $\mathbb{R}$. Prove that $\lambda(-H) = \lambda(H)$. (Hint: prove this for the Lebesgue outer measure by the definition.)

(54) ^{E} Let $f, g : \mathbb{R} \to \mathbb{R}^n$ be Borel-measurable functions. Prove that $h(x) = \langle f(x), g(x) \rangle$ is also Borel-measurable ($\langle \cdot, \cdot \rangle$ denotes the usual scalar product).

(55) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Prove that f is Borel measurable.

(56) ^{E} For any $t \in \mathbb{R}$ let δ_t denote the Dirac measure situated at t. Let $\mu = \delta_{-1} + \delta_1$, and $\nu = \delta_0 + \delta_2$. Consider the product measure $\mu \times \nu$ on $\mathbb{R}^2$. What is the value of $\mu \times \nu(D)$ for $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$?

(57) ^{E} Prove that for any $f, g \in L^1(X)$, we have $f + g \in L^1(X)$.

(58) ^{E} Calculate $\lim_{n\to\infty} \int_0^1 \frac{nx^n}{\cos x} dx$. (Hint: integrate by parts, and then LDCT.)

(59) Calculate $\lim_{n\to\infty} \int_1^\infty \frac{n}{1+n^2x^2} dx$, and $\lim_{n\to\infty} \int_0^\infty \frac{n}{1+n^2x^2} dx$. (Hint: in both cases you can actually evaluate the integrals by integrating by substitution. But please also check whether the LDCT can be used.)

(60) Calculate $\lim_{n\to\infty} \int_0^\infty \frac{1}{(1+\frac{x}{n})^n \sqrt[n]{x}} dx$. (Hint: calculate the pointwise limit for each $x > 0$, and then LDCT.)

(61) ^{E} Let $R = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 3, 0 \leq y \leq 2$ be a rectangle and $T = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 2x + y \leq 3\}$ a triangle in the plane. And let $f(x, y) = (x + y)^2 x$, and let λ_2 denote the Lebesgue measure on $\mathbb{R}^2$. Calculate $\int_R f(x, y) d\lambda_2$ and $\int_T f(x, y) d\lambda_2$. (Use Fubini, and please calculate both ways: $dxdy$ and $dydx$.)

(62) ^{E} Calculate: $\int_{x=0}^1 \int_{y=x}^1 x \frac{\sinh y}{y} dydx$. (Hint: notice that you get stuck in this order of integration, and use Fubini to change the order.)

(63) * Prove that $\int_0^\infty \frac{e^{-x}}{x} \sin x dx = \pi/4$. (Hint: apply Fubini to the function $e^{-xy} \sin x$.)

(64) ^{E} Let $f, g \in L^1(\mathbb{R})$. Prove that $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$. (This is basically trivial, and this is Minkowski's inequality for the exponent $p = 1$. For other exponents the proof is more difficult.)

(65) ^{E} Let $1 \leq p < r < +\infty$. Show examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \in L^p(\mathbb{R}), f \notin L^r(\mathbb{R})$, and $g \in L^r(\mathbb{R}), g \notin L^p(\mathbb{R})$. That is, $L^p(\mathbb{R})$ and $L^r(\mathbb{R})$ mutually do not contain each other. (Hint: restrict the function x^α to an appropriate domain, with an appropriate exponent α.)

(66) ^{E} Let $1 \leq p < r < +\infty$. Contrary to the previous exercise, prove that $L^r[0, 1] \subset L^p[0, 1]$. (Hint: for a function $f \in L^r[0, 1]$ split the interval $[0, 1]$ according to whether $f(x) \leq 1$ or $f(x) > 1$, and prove that on both parts the integral of $|f|^p$ is finite.)

(67) * The previous exercise is true in a stronger form. Let $\mu(X) < +\infty$. Prove that for any $1 \leq p < r < \infty$ we have $L^r(X) \subset L^p(X)$, and $\|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{r}} \|f\|_r$. (Hint: for the functions f and the constant 1 function use Hölder's inequality with appropriate exponents.)

(68) ^{E} Prove that the function $F(t) = \int_0^\infty e^{-tx} dx$ is differentiable infinitely many times on $(0, +\infty)$, calculate $F^{(n)}(1)$ and hence calculate the value of $\int_0^\infty x^n e^{-x} dx$. (Hint: We can differentiate under the integral sign.)

(69) Let $1 \leq p < +\infty$. Prove that the step functions are dense in $L^p(\mathbb{R})$. (Hint: for a nonnegative function $f \in L^p(\mathbb{R})$ approximate f from below with step functions, as in a theorem in a previous class. For the general case, take $f = f^+ - f^-$.)

Exercise sheet 9, Analysis 2, 2023

For the Fourier series of a function $f : [-\pi, \pi] \to \mathbb{R}$ we use the notation $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx$ for all $k \geq 1$, and $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$ for all $k \in \mathbb{Z}$. For a function $f \in L^1(\mathbb{R})$ we use the notation $\hat{f}$ for the Fourier transform of f, i.e. $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$.

(70) ^{E} Let $f(x) = 1+\cos x - 4\sin 2x$, for $-\pi \leq x < \pi$. Calculate the Fourier series of f. (Please calculate the real coefficients a_k, b_k, as well as the complex coefficients c_k). What do you observe? Calculate the value of $\int_{-\pi}^{\pi} |f(x)|^2 dx$.

(71) ^{E} Let $f(x) = |x|$, for $-\pi \leq x < \pi$. Calculate the Fourier series of f. (Only the coefficients a_k, b_k.) Prove that the Fourier series converges uniformly. Substitute $x = 0$ and hence calculate the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

(72) ^{E} Prove that if $f \in L^1(\mathbb{R})$, and $g(x) = f(x-a)$ then $\hat{g}(\xi) = e^{-ia\xi}\hat{f}(\xi)$. Also, if $h(x) = e^{iax}f(x)$ then $\hat{h}(\xi) = \hat{f}(\xi - a)$.

(73) ^{E} Prove that if $f \in L^1(\mathbb{R})$, and $g(x) = \frac{1}{\delta}f(\frac{x}{\delta})$ then $\hat{g}(\xi) = \hat{f}(\delta\xi)$. (Hint: direct computation.)

(74) Assume $f \in L^1(\mathbb{R})$ is continuously differentiable and $f' \in L^1(\mathbb{R})$. Prove that $\hat{f'}(\xi) = i\xi\hat{f}(\xi)$.

(75) Assume $f \in L^1(\mathbb{R})$, and $g(x) = xf(x) \in L^1(\mathbb{R})$. Prove that $\hat{g}(\xi) = i(\hat{f}(\xi))'$.

(76) Let $f, g \in L^1(\mathbb{R})$. Prove that $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

(77) ^{E} Let $f = 1_{[-a,a]}$ be the indicator function of the interval $[-a, a]$. Calculate $\hat{f}$.

(78) ^{E} Calculate the Fourier transform of $h(x) = e^{-|x|}$.

(79) Prove that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. (Hint: first calculate the integral $\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy$ with polar coordinates, then separate the variables.)

(80) Calculate the Fourier transform of $f(x) = e^{-x^2}$. (Hint: use the result of the previous exercise in a complex contour integral along an appropriate path.)

Exercise sheet 10, Analysis 2, 2023

(81) [E] Recall that the Schwartz space of function was defined as
$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall p, q \in \mathbb{N}, x^p D^q f(x) \to 0 \ (|x| \to \infty)\}$.
Prove that an equivalent definition would be the following:
$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall p, q \in \mathbb{N}, x^p D^q f(x) \ \textit{is bounded}\}$.
(Hint: if the limit was not 0 for some $p.q$ then, after multiplying by x, the function $x^{p+1} D^q f(x)$ would not be bounded.)

(82) Prove that for any $f \in S(\mathbb{R})$ and for any $p, q \in \mathbb{N}$ we have $\|D^q(x^p f(x))\|_1 < +\infty$. (Hint: we can estimate each term after differentiating multiple times.)

(83) [E] Let $f \in S(\mathbb{R})$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$. Prove that $\hat{\hat{f}}(x) = 2\pi f(-x)$. Prove that if f is real valued, then $\hat{f}$ is conjugate symmetric, i.e. $\hat{f}(\xi) = \overline{\hat{f}(-\xi)})$. And conversely, if f is conjugate symmetric, then $\hat{f}$ is real valued. Prove that if f is a real valued, even function then $\hat{f}$ is also real valued and even, and $\hat{\hat{f}} = 2\pi f$. (Hint: inversion formula).

(84) For any $a > 0$ calculate the Fourier transform of the following functions:
$f(x) = \frac{a}{x^2+a^2}$,
$f(x) = \left(\frac{\sin ax}{x}\right)^2$
(Hint: instead of using the definition, use the properties we have learnt about scaling and products.)

(85) [E] Let $f, g \in S(\mathbb{R})$. Prove that $f * g$ is differentiable and $(f * g)' = f' * g = f * g'$. (Hint: direct calculation.)

(86) Prove that $\int_{-\infty}^{\infty} \frac{\sin t \sin 2t}{t^2} dt = \pi$. (Hint: Parseval/Plancherel formula.)

(87) [E] Let $g(x) = e^{-x^2}$, and $g_\varepsilon(x) = \frac{1}{\varepsilon} g(\frac{x}{\varepsilon})$. Plot the graph of $g(x)$, $g_{0.01}(x)$, and $g_{100}(x)$.

(88) Let $f, g \in S(R)$, $g \geq 0$, $\int_{\mathbb{R}} g = 1$, and let $g_\varepsilon(x) = \frac{1}{\varepsilon} g(\frac{x}{\varepsilon})$. Prove that for every $x \in \mathbb{R}$ we have $f * g_e(x) \to f(x)$ as $\varepsilon \to 0$.

Exercise sheet 11, Analysis 2, 2023

(89) ^{E} Let $f, g \in L^1(\mathbb{R})$ be nonnegative functions. Show that $\|f*g\|_1 = \|f\|_1\|g\|_1$.

(90) ^{E} Let $f, g \in S(\mathbb{R})$. Show that $\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

(91) ^{E} Let $f(x) = 1_{[-a,a]}(x)$. Calculate $f*f(x)$.

(92) ^{E} Show that the Fourier transform is a linear operator, i.e. $\widehat{f+g} = \hat{f} + \hat{g}$, and $\widehat{\alpha f} = \alpha\hat{f}$ for all, $f, g \in L^1(\mathbb{R})$, and $\alpha \in \mathbb{R}$.

(93) Let $f_1(x) = 1_{[-1,1]}(x)$ and $f_2(x) = 1_{[-2,-1]}(x) + 1_{[1,2]}(x)$. Let $g(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$, and $g_\varepsilon(x) = \frac{1}{\varepsilon}g(\frac{x}{\varepsilon})$ for some small value of ε, like $\varepsilon = 0.01$. Try to plot the graph of $f_1 * g_e$ and $f_2 * g_e$. If you can use a software like Matlab or Mathematica to plot the graphs, please do so.

(94) ^{E} Let $f, g \in L^2(\mathbb{R})$. Prove that $|f*g(x)| \leq \|f\|_2\|g\|_2$ for all $x \in \mathbb{R}$. (Hint: Cauchy-Schwarz.)

(95) ^{E} Let $f \in L^1(\mathbb{R})$, and $|g(x)| \leq K$ for all $x \in \mathbb{R}$. Prove that $|f*g(x)| \leq K\|f\|_1$ for all $x \in \mathbb{R}$. (Hint: direct calculation.)

(96) * Prove that for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ we have $\left|(\frac{d}{dx})^n(\frac{\sin x}{x})\right| \leq \frac{1}{n+1}$. (Hint: Fourier transform both sides of the equation.)

(97) Let $f \in S(\mathbb{R})$. Solve the differential equation $u'' - u = f$. (Hint: Fourier transform both sides of the equation.)

(98) * Let $f \in L^2(\mathbb{R})$, $g \in S(R)$, $\int_{\mathbb{R}} g = 1$. Prove that $f*g(x)$ is finite for all $x \in \mathbb{R}$, $f*g$ is a smooth function, and for $g_\varepsilon(x) = \frac{1}{\varepsilon}g(\frac{x}{\varepsilon})$ we have $f*g_e \to f$ in L^2 norm as $\varepsilon \to 0$. (Moral of the story: the convolution has a smoothing effect, and the functions g_ε can be used to approximate any $f \in L^2(\mathbb{R})$.)

Exercise sheet 12, Analysis 2, 2023

For the Fourier series of a function $f : [-\pi, \pi] \to \mathbb{R}$ we use the notation $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx$ for all $k \geq 1$, and $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$ for all $k \in \mathbb{Z}$. Also, sometimes we use the notation $\hat{f}(k) = c_k$.

(99) Let $f, g \in S(\mathbb{R})$. Prove that $\widehat{f \cdot g}(\xi) = \frac{1}{2\pi}\hat{f} * \hat{g}(\xi)$.

(100) Let $f(x) = e^{-x^2} + e^{-|x|} + \frac{\sin x}{x}$, $g(x) = xe^{-|x|}$, and $h(x) = x^2 e^{-x^2}$. Calculate $\hat{f}, \hat{g}, \hat{h}$.

(101) ^{E} Let $f(x) = x$, for $-\pi \leq x < \pi$. Calculate the Fourier series of f. (Please calculate the complex coefficients c_k, as well as the real coefficients a_k, b_k.)

(102) ^{E} Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function. Show that $\sum_{k=-N}^{N} c_k e^{ikx} = a_0 + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx$, i.e. the partial sums of the Fourier series are exactly the same whether you consider the complex Fourier series or the real one.

(103) Prove the closed formula for the Dirichlet kernel: $D_N(x) = \sum_{k=-N}^{N} e^{ikx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}$.

(104) Plot the Dirichlet kernel for large N.

(105) * Prove that $\int_{-\pi}^{\pi} D_N(t)dt = 2\pi$ for all $N \in N$, but $\int_{-\pi}^{\pi} |D_N(t)|dt \to +\infty$ as $N \to \infty$.

(106) * Prove the closed formula for the Fejér kernel: $K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \frac{1}{n+1}(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})t})^2 = \frac{1}{n+1}(\frac{\sin\frac{(n+1)t}{2}}{\sin\frac{t}{2}})^2$.

(107) ^{E} Plot the Fejér kernel K_n for large n. Prove that $\int_{-\pi}^{\pi} K_n(t)dt = 2\pi$.

(108) ^{E} Let $f : \mathbb{R} \to \mathbb{C}$ be n-times continuously differentiable 2π-periodic function. Prove that for all $k \in \mathbb{Z}$ we have $\widehat{f^{(n)}(k)} = (ik)^n \hat{f}(k)$. (That is, we have a completely analogous formula as we had for Fourier transforms. The rule of thumb here: the smoother the function, the faster the Fourier coefficients converge to 0.)

(109) Let $f, g : \mathbb{R} \to \mathbb{C}$ be continuous 2π-periodic functions. Assume that $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. Show that $f = g$. (Hint: use Fejér's theorem.)

(110) ^{E} Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous 2π-periodic function and let $g(t) = f(t-y)$ for some fixed $y \in R$. Prove that $\hat{g}(k) = e^{-iky}\hat{f}(k)$. (That is, shifting the function will result in a phase factor on the Fourier side.)

Exercise sheet 13 (summary for midterm 2), Analysis 2, 2023

(111) Let $F(x) = \arctan(3x+1)$ if $x < 0$ and $F(x) = \frac{3x+1}{x+1}$ if $x \geq 0$. Let μ be the Lebesgue-Stieltjes measaure induced by F, i.e. $\mu((a,b]) = F(b) - F(a)$. Calculate $\mu([-1,0))$, $\mu([0,1])$ and $\int_{-1}^{2} 2 + x d\mu$.

(112) Let $f(x) = \frac{\sin x}{|x|^{3/2}}$, $g(x) = \frac{1-\cos x}{x^2}$. Decide whether $f, g \in L^1(\mathbb{R})$ or $L^2(\mathbb{R}$.

(113) Prove that if $f \in L^2(-\pi,\pi)$ then $f \in L^1(-\pi,\pi)$, and $\|f\|_1 \leq \sqrt{2\pi}\|f\|_2$.

(114) Let $1 \leq p \leq q \leq r < +\infty$. Prove that if $f \in L^p(\mathbb{R})$ and $f \in L^r(\mathbb{R})$, then $f \in L^q(\mathbb{R})$.

(115) Prove that if $f \in S(\mathbb{R})$, then for every $1 \leq p < +\infty$ we have $f \in L^p(\mathbb{R})$.

(116) Calculate the Fourier transform of the following functions:
$f(x) = \mathbf{1}_{[10,30]}$, $g(x) = xe^{-9x^2}$, $h(x) = \frac{1}{x^2+2x+2}$.

(117) Let $f \in S(\mathbb{R})$ be a real valued, even function, and let$g = f * f$. Prove that $\hat{g}$ is also a real valued, even function, and prove also that $\hat{g}(\xi) \geq 0$ for every $\xi \in \mathbb{R}$.

(118) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function. Prove that for every $a \in \mathbb{R}$ we have $\int_a^{a+2\pi} f(t)dt = \int_{-\pi}^{\pi} f(t)dt$.

(119) Let $g(x) = \sin^2(x)$, and $f(x) = -1$, ha $-\pi \leq x < 0$, and $f(x) = 1$, if $0 \leq x < \pi$. Calculate the Fourier series of g and f. (Both the complex coefficients c_n and the real coefficients a_n, b_n.)

(120) Let $f : \mathbb{R} \to \mathbb{R}$ be a π-periodic function. Prove that in the Fourier series of f all the coefficients with odd indices are zero.

(121) Let $f \in L^2[-\pi,\pi]$. Prove that $|\hat{f}(k)| \leq \frac{1}{\sqrt{k}}$ is satisfied for infinitely many indices k. (Hint: Parseval.)

(122) * Let $f : [0,\pi] \to \mathbb{R}$ be continuously differentiable and $f(0) = f(1) = 0$. Prove that $\int_0^{\pi} |f(t)|^2 \leq \int_0^{\pi} |f'(t)|^2$. (This is the Wirtinger-inequality.)