Practise exercises 1.

(1) Prove that $\|\mathbf{x}-\mathbf{y}\| \geq |\|\mathbf{x}\| - \|\mathbf{y}\||$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$.

(2) Let $K > 0$ be an arbitrary fixed positive number. Prove that $\mathbf{x}_n \to \mathbf{a}$ if and only if for every $\varepsilon > 0$ there exists N such that for every $n \geq N$ we have $\|\mathbf{x}_n - \mathbf{a}\| < K\varepsilon$.

(3) Prove that for any $A \subset \mathbb{R}^p$ we have $int(A) \cup \partial(A) \cup ext(A) = \mathbb{R}^p$, and the unions are disjoint.

(4) Consider $\mathbb{Q} \subset \mathbb{R}$. Find $int(Q), \partial(Q), ext(\mathbb{Q})$.

(5) Prove that $int(A)$ is open for any set $A \subset \mathbb{R}^p$.

(6) Prove that ∂A is closed for any $A \subset \mathbb{R}^p$.

(7) Let C denote the Cantor set (defined by always removing the middle third of the remaining intervals, as in class). Prove that $int(C) = \emptyset$.

(8) Prove that if $A \neq \emptyset, \mathbb{R}^p$ then A cannot be open and closed at the same time.

(9) Let $A_2 = \{(x, y) \in \mathbb{R}^2 : x = 1/n(n = 1, 2, \dots), y \in (0, 1)\}$. Draw a picture of A_2 and find $int(A_2)$ and $\partial(A_2)$.

(10) Prove that if $\mathbf{x}_n \to \mathbf{a} \in \mathbb{R}^p$ then $\langle \mathbf{x}_n, \mathbf{b} \rangle \to \langle \mathbf{a}, \mathbf{b} \rangle$.

(11) Find the following limits, if they exist:

$\lim_{(x,y)\to(2,3)} \frac{x-2}{y-3}$

$\lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{x^2y^5}}{x^2+y^2}$

$\lim_{(x,y)\to(0,0)} \frac{\sin x - \sin y}{x-y}$

$\lim_{(x,y)\to(0,0)} (1+x)^y$

$\lim_{(x,y)\to(1,1)} \frac{xy-1}{x-1}$

Practise exercises 2.

(12) Let $A = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 2, 0 < y < x^2\}$ and $B = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = \sin \frac{1}{x}\}$. Determine $intA, intB, \partial A, \partial B$.

(13) Prove that $G = \{(x, y) \in \mathbb{R}^2 : 0 < y, x^2 + y^2 < 1\}$ is open, and $F = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, 0 \leq y, z \leq e^{x+y}\}$ is closed.

(14) Find the following limits, if they exist:

$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{\sqrt{x^2+y^2}}$

$\lim_{(x,y)\to(0,0)} \frac{x-2y}{3x+y}$

$\lim_{(x,y)\to(1,0)} \frac{\ln(x+e^y)}{x^3+y^3}$

(15) Prove that if $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ and $\mathbf{x} \perp \mathbf{y}$ (i.e. $\langle \mathbf{x}, \mathbf{y} \rangle = 0$) then $\|\mathbf{x}-\mathbf{y}\| = \sqrt{2}$.

(16) Prove that $(\mathbf{x}-\mathbf{y}) \perp (\mathbf{x}+\mathbf{y})$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

(17) Prove that if $\mathbf{x}_n \to \mathbf{a}$ and $\mathbf{y}_n \to \mathbf{b}$ then $\langle \mathbf{x}_n, \mathbf{y}_n \rangle \to \langle \mathbf{a}, \mathbf{b} \rangle$.

(18) Calculate the partial derivatives of the following functions:

$$f(x,y) = \sin xy + xy^2 - \ln(x+y)$$

$$f(x,y) = e^{-1/(x^2+y^2)}$$

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

$$f(x,y,z) = \arctan(x^2+y^4+z^6+1)$$

(19) Let $f(x,y,z) = x^3 + y^4 + x^2 y e^{2z}$. Calculate all partial derivatives of f up to order 2 (i.e. all expressions like $\frac{\partial f^2}{\partial x \partial y}$, $\frac{\partial f^2}{\partial x \partial z}$, etc).

(20) Consider the surface given by $z = \frac{16}{xy}$. Give the equation of the tangent plane at the point $(x,y,z) = (1,2,8)$.

(21) Let $f(x,y) = x^2 - xy + 3y^2$, $(x_0,y_0) = (1,2)$, $\mathbf{u} = (-3,4)$. Give the directional derivative of f at (x_0,y_0) in the direction of $\mathbf{u}$. (Normalize $\mathbf{u}$ first!)

Practise exercises 3.

(22) The equation $z = x^2y + xy^2 + x + 3y - 1$ defines a landscape, and at the point $(4,1,26)$ of this landscape there is a spring. In which direction will the water flow from the spring?

(23) Give the second order Taylor polynomial of the following functions at the given points:

$$f(x,y) = \frac{x}{y} \text{ at } (1,2)$$

$$f(x,y,z) = x^3 + y^3 + z^3 \text{ at } (1,2,3)$$

$$f(x,y) = \sin(x+2y) \text{ at } (\pi/4, \pi/6).$$

(24) Find the local maxima and local minima of the following functions:

$$f(x,y) = x^2 + xy + y^2 - 3x - 3y$$

$$f(x,y) = x^3y^2(2-x-y)$$

$$f(x,y) = x^3 + y^3 - 9xy$$

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

(25) Find the maximal region in the plane where the function $f(x,y) = x^3 + y^3 - 9xy$ is convex.

(26) Find the derivative (i.e. the Jacobian matrix) of the following functions:

$$f(x,y) = (x^2y, x+y, ye^x)$$

$$f(x,y) = (\sin(x-3y), \ln(x+y))$$

$$f(x,y,z) = (x+y^2+z^3, z^4\sin(ye^x))$$

(27) Let $f : \mathbb{R}^p \to \mathbb{R}$ be differentiable, and let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$. Let $F(t) = f(\mathbf{a}+t(\mathbf{b}-\mathbf{a})$ for $t \in [0,1]$. Prove that $F'(t) = \langle f'(\mathbf{a}+t(\mathbf{b}-\mathbf{a}), \mathbf{b}-\mathbf{a}\rangle$ for every $t \in [0,1]$. Prove also that this implies that there exist a $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ such that $f(\mathbf{b}) - f(\mathbf{a}) = \langle f'(\mathbf{c}), \mathbf{b}-\mathbf{a}\rangle$. (This is the Mean Value Theorem.)

Practise exercises 4.

(28) Chain Rule: Let $f(t) = \left(t^2 - t, \frac{1}{1+t^2}, e^t\right)$, $g(x,y,z) = x^2y - z$, and $t_0 = 1$, $\underline{a} = (1,2,3)$. Determine $(f \circ g)'(\underline{a})$ and $(g \circ f)'(t_0)$ by applying the chain rule.

(29) Inverse functions: let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(u,v) = (u^3 + uv + v^3, u^2 - v^2)$. Plug in $(u,v) = (1,1)$ to obtain $\varphi(1,1) = (3,0)$ Show that in a small neighbourhood of $(3,0)$ the inverse function φ^{-1} exists and determine the derivative $((\varphi^{-1})'((3,0)$.

(30) Lagrange multipliers: determine the local maxima and minima of the following functions under the following constraints:

$$f((x,y,z) = x - y + 3z,\ x^2 + y^2/2 + z^2/3 = 1;$$

$$f(x,y,z) = x^2 + y^2 + z^2,\ x + 2y + z = 1,\ 2x - y - 3z = 4;$$

$$f(x,y,z,t) = x^2 + 2y^2 + z^2 + t^2,\ x + 3y - z + t = 2,\ 2x - y + z + 2t = 4.$$

(31) Implicit differentiation: consider the equation $x^2y + 3x^3z^2 - xyz + \ln(2x + y - z) - 23 = 0$. Show that the point $(1,2,3)$ satisfies the equation. Determine the derivatives $\partial z/\partial y$, $\partial z/\partial x$ and $\partial y/\partial x$ at this point.

Practise exercises 5.

(32) Chain rule: let $r(\mathbf{x}) = r(x,y,z) = \sqrt{x^2+y^2+z^2}$. Differentiate $r^5(x,y,z)$ in two ways: first write out $r^5(x,y,z) = (x^2+y^2+z^2)^{5/2}$ and calculate its partial derivatives one by one. Second, use the chain rule for the functions $f(t) = t^{5/2}$ and $g(x,y,z) = x^2+y^2+z^2$. (Of course, you should get the same result.)

(33) Reminder: the product rule for derivatives of scalar functions $f, g : \mathbb{R}^p \to \mathbb{R}$ is given by $(f \cdot g)' = gf' + fg'$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be a fixed vectors, and consider the inner product with $\mathbf{u}$, $f(\mathbf{x}) = \langle \mathbf{u}, \mathbf{x}\rangle$. First show that $f'(\mathbf{x}) = \mathbf{u}$ for all $\mathbf{x} \in \mathbb{R}^3$. Also, let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $h(\mathbf{x}) = \langle \mathbf{u}, \mathbf{x}\rangle\langle \mathbf{v}, \mathbf{x}\rangle$. Show that $h'(\mathbf{x}) = \langle \mathbf{u}, \mathbf{x}\rangle\mathbf{v} + \langle \mathbf{v}, \mathbf{x}\rangle\mathbf{u}$.

(34) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + x^3 + \cos x$. Prove that $f^{-1} : \mathbb{R} \to \mathbb{R}$ exists on the whole real line, f^{-1} is differentiable, and $(f^{-1})'(1) = 1$.

(35) Lagrange multipliers. Determine the local maxima and minima of the following function with the given constraints: $f(x,y,z) = x^2 + y^2 - x - z$, $x + y + z = 0$, $2x^2 - y + z = 0$.

(36) Implicit differentiation. Consider the equations

$$x_1^3 y_2 + x_1 x_3 + 3x_1 x_2 y_1^2 + x_3 y_1 y_2^2 = 3$$
$$x_1^2 x_2 y_1 - x_3 y_1^2 y_2 + 7x_2 y_2^5 = -5$$

Show that the point $(0, 2, 1, 3, -1)$ satisfies these equations. Show that in a small neighbourhood of the point $(0, 2, 1)$ we can express the variables y_1, y_2 as a function $(y_1, y_2) = \varphi(x_1, x_2, x_3)$ and determine the derivative matrix $\varphi'(0, 2, 1)$.

(37) Repeat the calculation we did in class to calculate the volume of the k-dimensional unit ball: $\gamma_{2k} = \frac{\pi^k}{k!}$ and $\gamma_{2k+1} = \frac{\pi^k 2^{2k+1} k!}{(2k+1)!}$.

(38) Assume $H \subset \mathbb{R}^{p-1}$ is a convex set. Consider the cone C determined by a point $(z_1, \ldots, z_p) \in \mathbb{R}^p$ (where $z_n > 0$) and H embedded in the coordinate-hyperplane $x_n = 0$. Show that this cone is also convex. As all convex sets are Jordan measurable, this shows that C is Jordan measurable. Next, by taking horizontal slices, show that the measure of C is $t^p(C) = \frac{1}{p} t^{p-1}(H) z_p$.

(39) Draw a picture and calculate the area of the region $T = \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x^2 \leq y\}$.

(40) Let T be the trapezoid in $\mathbb{R}^2$ given by the points $(0, 0), (3, 0), (2, 1), (1, 1)$, and let $f(x, y) = 3x - y^2$. Calculate $\int_T f$.

(41) Polar coordinates. Let $T = \{(x, y) \in \mathbb{R}^2 : 9x^2 + \frac{y^2}{4} \leq 1, x \geq 0\}$, and $f(x, y) = 1 + \sqrt{9x^2 + \frac{y^2}{4}}$. Calculate $\int_T f$

Practise exercises 6: practice for midterm exam I.

(42) Prove that if $\mathbf{x} \perp \mathbf{y}$, then for all $\alpha \in \mathbb{R}$ we have $\|\mathbf{x} + \alpha \mathbf{y}\| \geq \|\mathbf{x}\|$.

(43) Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 < 9, x + y \geq 1\}$. Draw a picture of A and determine $int(A)$ and $\partial(A)$.

(44) Find the following limits if they exist:

$$\lim_{(x,y)\to(0,0)} \frac{y^3 - 3x^2 y}{x^2 + xy + y^2}$$
$$\lim_{(x,y)\to(0,0)} \frac{4x^2 - 2xy + 3y^2}{x^2 - xy + 2y^2}$$

(45) Consider the surface given by the equation $z = 4x^2 y + xy^3 + \ln(2x - y)$. Show that the point $(x, y, z) = (1, 1, 5)$ is on the surface. Find the equation of the tangent plane at this point. Also, find the directional derivative at this point in the direction $\mathbf{u} = (-5, 12)$.

(46) Give the second order Taylor polynomial of the function x^y at the point $(2, 3)$ and give an estimate of $2.01^{2.98}$ accordingly.

(47) Find the local minima and maxima of the following functions:

$$f(x, y) = x^3 + y^3 - 3xy$$
$$f(x, y) = 4x^2 + 2xy - 5y^2 + 2$$

(48) Find the maxima and minima of the following function under the given constraints:

$$f(x,y,z) = x^2 + y^2 - x - z, 2x + y + z = 0, 2x^2 - y + z = 0$$

(49) Chain rule: let $f(u,v) = \sqrt{u^2+v^2}$ and $z(x,y) = f(xe^y, x^2 - 3y)$. Determine $\partial z/\partial x$ and $\partial z/\partial dy$.

(50) Implicit differentiation: consider the equation $z\sin(x-y) + 3x^2yz + \ln(x + 2y - z) = 6$. Show that the point $(1,1,2)$ satisfies this equation, and determine the value of $\partial z/\partial x$ and $\partial z/\partial dy$ at this point.

(51) Inverse functions: let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(u,v) = (2u^3v + uv + uv^2, u\ln(3u - v) + 3v^2)$. Plug in $(u,v) = (1,2)$ to obtain $\varphi(1,2) = (10,12)$ Show that in a small neighbourhood of $(10,12)$ the inverse function φ^{-1} exists and determine the derivative $((\varphi^{-1})'((10,12)$.

(52) Draw a picture and calculate the area of the region $T = \{(x,y) \in R^2 : x^2 + y^2 \leq 1, y \geq 0, x \leq y, x^2 \leq y\}$.

(53) Let T be the triangle given given by the points $(0,0), (2,0), (1,1)$, and let $f(x,y) = 3x^2 - y + xy$. Calculate $\int_T f(x,y)dxdy$.

(54) Polar coordinates. Let $T = \{(x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} \leq 1, y \geq 0\}$, and $f(x,y) = x + 3y$. Calculate $\int_T f$.

Practise exercises 7.

(55) Cylindrical coordinates: let $f(x,y,y) = z^2$ and $V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \geq 1, x^2 + y^2 + z^2 \leq 4\}$. Calculate $\int_V f$.

(56) Cylindrical coordinates: let $f(x,y,z) = 2ye^{x^2+z^2}$ and $V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + z^2 \leq 1, 0 \leq y \leq 1\}$.

(57) Spherical coordinates: determine the center of gravity of the half-ball $V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0\}$.

(58) Calculate the arc-length of the curve $r(t) = (t+1, \frac{t^2}{2}, \frac{2\sqrt{2t^3}}{3})$, $t \in [-2,0]$.

(59) Calculate the arc-length of the curve $r(t) = (\cos t - t\sin t, \sin t - t\cos t, t)$, $t \in [0,1]$.

(60) Normal vector and tangent plane at a point of a surface: let $r(u,v) = (u^2 - 2v^2, uv - v^3, u^4 - 2v)$ describe a surface. Calculate the normal vector to the surface at the point $(u,v) = (-1,1)$, and give the equation of the tangent plane.

(61) Let $r(u,v) = (u+v, u^2+v^2, u^3+v^3)$. Calculate the normal vector at the point $(u,v) = (1,-1)$ and give the equation of the tangent plane.

(62) Calculate the surface area of the following surface: $r(u,v) = (u^2, 2u\cos v, 2u\sin v,$ $u \in [0,1], v \in [0,\pi/2]$.

(63) Calculate the surface area of the following surface: $r(u,v) = (u\cos v, u\sin v, v)$, $u \in [-1,1], v \in [0,2\pi]$. Try to visualize the surface.

(64) Integration along a curve: a curve G is given by $r(t) = (t, t^2, t^3)$, $0 \le t \le 2$, and a vector field $F(x, y, z) = (y^2 - x^2, 2yz, -x^2)$ is given. Calculate $\int_G F(r)dr$.

(65) Let $A = (1, -2, 3)$, $B = (2, 1, 4)$, and consider the line segment L joining A and B. Also, let $F(x, y, z) = (y + z, x + z, x + y)$. Calculate $\int_L F$.

(66) Surface integral: let $F(x, y, z) = (x, y, z)$ and let the surface S be given by $r(u, v) = (3\cos v, 3\cos u \sin v, \sin u)$, $u \in [0, \pi]$, $v \in [0, 2\pi]$. Calculate $\int F dS$.

(67) Let $F(x, y, z) = (xy, 0, 2x + z)$ and let S be given by $r(u, v) = (u + 2v, -v, u^2 + 3v)$, $0 \le u \le 3$, $-2 \le v \le 0$. Calculate $\int F dS$.

Practise exercises 6: practice for midterm exam II.

(68) Cylindrical coordinates: let $f(x, y, z) = x + y + z$, and $V = \{(x, y, z) : x^2 + y^2 \le 1, z \ge 0, z \le 4 + x + 2y\}$. Calculate $\int_V f$.

(69) Spherical coordinates: let $f(x, y, z) = 2x + y + 1$ and let $V = \{(x, y, z) : (x - 1)^2 + y^2 + z^2 \le 4\}$. Calculate $\int_V f$.

(70) Arc-length: calculate the arc-length of the following curve:
$r(t) = (t, \sqrt{4t - t^2}, 2\ln(1 - \frac{t}{4}))$, $0 \le t \le 1$.

(71) Calculate the arc-length of the parabola $y = x^2$ where x ranges from 0 to 1.

(72) Normal vector and tangent plane: calculate the normal vector and tangent plane of the given surface at the given point. The surface is $r(u, v) = (uv, u^2 + 3v^2, uv^2 - 1)$ and the point is $(u, v) = (1, 2)$.

(73) Calculate the surface area of the following surface:
$r(u, v) = (\cos u - v \sin u, \sin u + v \cos u, u + v)$, $0 \le u \le \pi$, $0 \le v \le 1$.

(74) Calculate the surface area of the following surface:
$S = \{(x, y, z) : x^2 + y^2 \le 1, z = xy\}$.

(75) Integration along a curve: let $F(x, y, z) = (xy, yz, 1)$ and let a curve be given by $r(t) = (t^2, 2t + 1, t^3)$, where $0 \le t \le 3$, Calculate $\int_G F(r)dr$.

(76) Integration along a curve: let $F(x, y, z) = (x, y, z)$ and let G be the unit circle (with anti-clockwise orientation) in the $z = 0$ plane. Calculate $\int_G F$.

(77) Integration along a surface: let $F(x, y, z) = (\frac{1}{xz}, \frac{1}{yz}, 0)$, and a surface S be given by $r(u, v) = (\cos^3 u \cos v, \cos^3 u \sin v, \sin^3 u)$, where $\pi/4 \le u \le \pi/2$, $0 \le v \le \pi/2$. Calculate $\int F dS$.

(78) Divergence theorem: let $F(x, y, z) = (2x + e^{y \cos z}, y^2 + \sin(2x) \arctan z, 2z)$, and let S be the surface of the half-sphere determined by $x^2 + y^2 + z^2 = 1$, $z \ge 0$ (with the normal vector of the surface pointing outside). Use the divergence theorem to determine $\int F dS$.

(79) Scalar potential. Let $F(x, y, z) = (2xy - yz, x^2 + 3y^2 z - xz, y^3 - xy)$. Prove that F has a scalar potential $v(x, y, z)$ and calculate v.

(80) Divergence and curl: let $F(x, y, z) = (x^2 + y^3, 12xy - 3x, xyz^2)$. Calculate the divergence and rotation (curl) of F at the point $(x, y, z) = (1, 3, 5)$.

(81) Stokes theorem. Let $H(x,y,z) = (xz^2, -yz^2, 3(4y-1-y^2)$. By solving the previous exercise you can easily determine a vector potential F of H. With the help of this vector potential calculate the surface integral $\int_H dS$, where the surface S is determined by $r(u,v) = (u\cos v, u\sin v, 1+\frac{\pi}{4} - \arctan u)$, where $0 \le u \le 1$, $0 \le v \le 2\pi$.

(82) What is the point-wise limit of the function sequence $f_n(x) = \frac{1-(\ln x)^n}{1+(\ln x)^n}$ where $1 \le x$? Is the convergence uniform?

(83) Let $f_n(x) = \frac{1}{x^2+n^{1/n}}$, where $0 \le x \le 1$. Calculate $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

(84) Use the Weierstrass criterion to prove that $\sum_{n=1}^{\infty} \frac{1}{n} e^{-nx^2}$ converges uniformly on the set $H = [1, +\infty)$.

(85) Use the Cauchy criterion to prove that $\sum_{n=1}^{\infty} \frac{1}{n} e^{-nx^2}$ does not converge uniformly on the set $H = (0, +\infty)$.

(86) Determine the radius of convergence and calculate the sum of the following power series:

$f(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$,

$g(x) = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{3^{n+1}}$,

$h(x) = \sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{n+1}$

(87) Let $f(x) = 1$ if $-\pi \le x \le 0$ and $f(x) = x$ if $0 < x < \pi$, and extend f 2π-periodically to $\mathbb{R}$. Calculate the Fourier coefficients of f.

(88) Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, continuously differentiable function. Prove that the Fourier coefficients of f and f' satisfy the following equalities: $a_0(f') = 0$, $a_n(f') = nb_n(f)$ and $b_n(f') = -na_n(f)$.

(89) Use the formula $e^{ix} = \cos x + i \sin x$ to calculate the following Fourier series in closed form: $\sum_{n=1}^{\infty} q^n \cos nx$ for $|q| < 1$.

(90) First calculate the sum for the series $\sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}$. Then use the formula $e^{ix} = \cos x + i \sin x$ and to evaluate the Fourier series $\sum_{n=2}^{\infty} \frac{\cos nx}{n(n-1)}$ in closed form.