FUN Exercise sheet 1.

(1) On $\mathbb{R}$, does either of the functions $d(x,y) = (x-y)^2$ or $d(x,y) = \sqrt{|x-y|}$ define a metric?

(2) Prove that $(\mathbb{K}^n, \|\cdot\|_2)$ is a normed space, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $\|\cdot\|_2$ denotes the Euclidean norm, $\|\mathbf{x}\| = \sqrt{\sum_{j=1}^n |x_j|^2}$.

(3) Prove that the space of continuous functions $C[a,b]$ is a normed space with the norm $\|\mathbf{x}\| = \max_{t\in[a,b]} |\mathbf{x}(t)|$.

(4) Give an example of a metric space in which there exists a small ball of radius r_1 which contains (and not equal to) a bigger ball with radius $r_2 > r_1$.

(5) Prove that the situation of the previous exercise cannot happen in a normed space (over $\mathbb{R}$ or $\mathbb{C}$).

(6) Prove that $B(x_0,r) = \{x \in X : d(x,x_0) < r\}$ is open, and $\overline{B}(x_0,r) = \{x \in X : d(x,x_0) \leq r\}$ is closed.

(7) (HW1) Prove that $\mathrm{Int}(M)$ is open for any subset $M \subset X$ of any metric space (X,d).

(8) Prove that in any metric space, every convergent sequence is Cauchy, and every Cauchy sequence is bounded.

(9) Two distance functions d_1 and d_2 on the same set X are called equivalent if there exist positive constants $a,b > 0$ such that $ad_1(x,y) \leq d_2(x,y) \leq bd_1(x,y)$. Prove that a sequence is convergent in (X,d_1) if and only if it is convergent in (X,d_2).

(10) (HW2) Prove that $C[a,b]$ is a normed space with the integral norm $\|\mathbf{x}\| = \int_{t=a}^{b} |x(t)|dt$.

(11) Let (X_1,d_1) and $(X_2,d_2$ be metric spaces. Let $X = X_1 \times X_2$ and $d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2)$. Prove that (X,d) is a metric space.

(12) Prove that l^p is separable for $1 < p < \infty$.

(13) (HW3) Find a sequence $(a_n) \in l^p$ for all $1 < p \leq \infty$ but $(a_n) \notin l^1$. Also, find a sequence $a_n \to 0$ but $(a_n) \notin l^p$ for any $1 \leq p < \infty$.

(14) Prove that in a discrete metric space every subset is open *and* closed.