FUN Exercise sheet 2. – date of submission 2018.02.27.

(15) Prove that for $0 < p < 1$ the formula $||x||_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$ does not define a norm in the space $\left\{x \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$.

(16) Let $\mathbf{x}(t) = t^2$. Visualize the open ball $B(\mathbf{x}, 1/2)$ in $(C[-1,1], \|\cdot\|_\infty)$.

(17) Prove that $(l^\infty, \|\cdot\|_\infty$ is complete.

(18) (HW1) Prove that $(C[a,b], \|\cdot\|_\infty)$ is complete.

(19) (HW2) Let $M \subset l^\infty$ denote the set of sequences which contain only finitely many non-zero terms. Show that M is not complete (with the metric inherited from l^∞).

(20) Prove that $|\|y\| - \|x\|| \leq \|y - x\|$.

(21) What do the unit balls look like in 2-dimensions for the l^p norm, as p ranges from 1 to ∞?

(22) Prove that the closed unit ball $\overline{B}(0,1)$ is convex in any normed space.

(23) (HW3) Prove that the closure of a convex set is convex in any normed space.

(24) Show that the vector space operations $(x, y) \mapsto x + y$ and $x \mapsto \alpha x$ are continuous in any normed space.

(25) Prove that in a Banach space, an absolutely convergent series is convergent.

(26) Show an example of a normed space and a series which is absolutely convergent but not convergent.

(27) Prove that a non-trivial subset of a normed space cannot be open and closed at the same time.