FUN Exercise sheet 3. – date of submission 2018.03.01.

(28) Let X be a normed space, and $Y \subset X$ a closed subspace. Prove that the quotient space X/Y becomes a normed space with the norm $\|[x]\| = \inf_{y\in Y} \|x+y\|$.

(29) Prove that $\mathrm{Ran}(T)$ and $\mathrm{Ker}(T)$ are vector spaces for any linear operator T, and if $\dim(\mathrm{Dom}(T)) = n < \infty$, then $\dim(\mathrm{Ran}(T)) \leq n$.

(30) (HW1) Let $1 \leq p < r \leq +\infty$, and consider the spaces $L^p(\mathbb{R})$ and $L^r(\mathbb{R})$. Does either of them contain the other?

(31) Let X be the space of real-valued functions on $\mathbb{R}$ whose derivatives of all orders exist everywhere on $\mathbb{R}$. Let $T : X \to X$ be defined by $x(t) \mapsto x'(t)$. Show that T is linear and surjective but T^{-1} does not exist.

(32) Let X be the space of polynomials on $[0,1]$ with the $\|\cdot\|_{\max}$ norm. Show that differentiation, $x(t) \mapsto x'(t)$ is a linear operator on X, but it is not bounded.

(33) (HW2) Let $f \in C[0,1]$ be given, and let $M_f : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ be the multiplication operator by f, i.e., $\forall g \in C[0,1]$, $(M_f g)(x) := f(x)g(x)$. Prove that M_f is a linear operator and determine its norm.

(34) Let $X = (C[0,1], \|\cdot\|_{\max})$, and let $T : X \to X$ be defined by $Tx(t) = \int_0^t x(\tau)d\tau$. Show that T is bounded linear, and determine its norm $\|T\|$.

(35) For any linear operators $T : X \to X$, $T_1 : Y \to Z$ and $T_2 : X \to Y$ we have $\|T_1T_2\| \leq \|T_1\|\|T_2\|$, and $\|T^n\| \leq \|T\|^n$.

(36) Show that a linear operator $T : X \to Y$ is bounded iff it maps bounded sets to bounded sets.

(37) Show that $T : l^\infty \to l^\infty$ given by $(x_1, \ldots x_j, \ldots) \mapsto (x_1/1, \ldots, x_j/j, \ldots)$ is bounded linear.

(38) Show that the range of a bounded linear operator $T : X \to Y$ need not be closed in Y.

(39) Show that the inverse of a bounded linear operator need not be bounded.

(40) (HW3) Let $T : X \to Y$ be a surjective bounded linear operator. Assume that there exist a $b > 0$ such that $\|Tx\| \geq b\|x\|$ for all $x \in X$. Prove that T^{-1} exists and is bounded.

(41) Let X be the space of bounded real-valued functions on $\mathbb{R}$ with the sup-norm. Define $T : X \to X$ by $x(t) \mapsto x(t-1)$. Prove that T is bounded linear and determine its norm.