FUN Exercise sheet 4. – date of submission 2018.03.08.

(42) (HW1) Find the norm of the linear functional f defined on $C[-1,1]$ by $f(\mathbf{x}) = \int_{-1}^{0} \mathbf{x}(t)dt - \int_0^1 \mathbf{x}(t)dt$.

(43) Let $C^1[0,1]$ denote the space of continuously differentiable functions on $[0,1]$. Introduce the norm $\|\mathbf{x}\| = \max_{t\in[0,1]} |\mathbf{x}(t)| + \max_{t\in[0,1]} |\mathbf{x}'(t)|$. Show that it is indeed a norm. Show that the functional $f(\mathbf{x}) = \mathbf{x}'(\frac{1}{2})$ is linear and bounded, and determine its norm.

(44) Give an example of two norms on the same vector space, $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ such that the identity mapping $I : X \to X$ is not continuous!

(45) Given a closed, convex symmetric body A in $\mathbb{R}^3$, prove that there exists a norm $\|\cdot\|$ on $\mathbb{R}^3$ such that $A = \overline{B}(0,1)$.

(46) If f is a bounded linear functional on a complex normed space, then is $\overline{f}$ also bounded and linear? (The bar is complex conjugation.)

(47) Let f be a linear functional on a normed space X. Prove that f is bounded iff $Kerf$ is a closed subspace. Prove that if $Kerf$ is not closed then it is dense in X.

(48) (HW2) A linear functional f on $X = C[0,1]$ is called positive if $f(x) \geq 0$ for all nonnegative functions $x(t)$. Prove that $f \in X'$.

(49) Prove that if X is a compact metric space, and $M \subset X$ is closed then M is compact.

(50) Let X, Y be metric spaces, X compact, and $f : X \to Y$ be a continuous bijection. Prove that f^{-1} is also continuous.

(51) Let $\dim X = n < \infty$ and Y be any normed space. Let $T : X \to Y$ be a linear operator. Show that T is automatically bounded.

(52) (HW3) Let $\dim X = n < \infty$. Show that $X' = X^*$ and $\dim X' = n$.

(53) Give an example of a bounded linear operator $T : X \to X$ such that $\overline{T(B(0,1))}$ is compact but $\mathrm{Ran}T$ is not finite dimensional.

(54) Let c_0 denote the vector space of real sequences converging to zero, with the sup norm. Show that the dual space of c_0 is l^1.

(55) Prove that c_0 is not reflexive.

(56) Define a linear functional f on c_0 by $(x_1, x_2, \dots, x_n, \dots) \mapsto \sum_{n=1}^{\infty} \frac{x_n}{n^2}$. What is the norm of f?