FUN Exercise sheet 5. – date of submission 2018.03.20.

(59) Prove the parallelogram law in Hilbert spaces: $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$.

(60) Give an example of a normed space where the norm cannot be induced by an inner product.

(61) Prove the polarization identities in Hilbert spaces:
over $\mathbb{R}$ we have: $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$

over $\mathbb{C}$ we have: $\mathrm{Re}\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$, and $\mathrm{Im}\langle x, y\rangle = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2)$.

(62) Prove the continuity of the inner product: if $x_n \to x$ and $y_n \to y$ then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

(63) Prove that if $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ then $x_n \to x$.

(64) (HW1) Show that in a Hilbert space x is orthogonal to y if and only if $\|x+\alpha y\| = \|x-\alpha y\|$ for all $\alpha \in \mathbb{K}$.

(65) We proved that for every closed subspace Y in a Hilbert space H there exists a projection operator $P_Y : H \to H$ which assigns to every vector $x \in H$ the vector $y \in Y$ which is closest to x. Prove that P_Y is a bounded linear operator.

(66) Let $Y \subset l^2$ be the subspace $Y = \{\mathbf{x} = (x_1, x_2, \dots) \in l^2 : x_{2n} = 0 \text{ for every } n \in \mathbb{N}\}$. Show that Y is a closed subspace and find $Y^\perp$.

(67) (HW2) Let $M \subset H$ be an arbitrary non-empty subset of a Hilbert space H. Prove that $M^\perp$ is a closed subspace.

(68) Prove that an orthonormal set is linearly independent.

(69) (HW3) Let (e_k) be an orthonormal basis in H, and let x, y be arbitrary vectors. Prove that

$$\langle x, y\rangle = \sum_{k=1}^{\infty} \langle x, e_k\rangle \overline{\langle y, e_k\rangle}$$

.

(70) Let $H = L^2(-1, 1)$ and let $f_1(t) = 1, f_2(t) = t, f_3(t) = t^2 + at + b$. Choose the parameters a, b so that f_3 be orthogonal to f_1, f_2.

(71) Let $H = L^2(-\pi, \pi)$, and let $f(t) = t$. Calculate the real Fourier series of f. Hence determine the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$.