FUN Exercise sheet 6. – date of submission 2018.04.10.

(72) Prove the properties of the adjoint operator:
 (a) $\langle T^*y, x\rangle = \langle y, Tx\rangle$
 (b) $(S + T)^* = S^* + T^*$
 (c) $(\alpha T)^* = \overline{\alpha} T^*$
 (d) $(T^*)^* = T$
 (e) $\|T^*T\| = \|TT^*\| = \|T\|^2$
 (f) $T^*T = 0$ iff $T = 0$
 (g) $(ST)^* = T^*S^*$.

(73) Prove that if $T_n \to T$ then $T_n^* \to T^*$.

(74) Prove that if $T : H \to H$ is bijective and bounded, then $(T^{-1})^* = (T^*)^{-1}$.

(75) (HW1) Let $T : l^2 \to l^2$ be the right shift operator, i.e. $T(x_1, x_2, \dots,) = (0, x_1, x_2, \dots)$. Find T^*. Show that $T^*T = I$ but T is not unitary.

(76) Show that for any bounded linear operator $T : H \to H$ the operators $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$ are both self-adjoint, and $T = T_1 + iT_2$.

(77) (HW2) Let $T : H \to H$ be a non-zero bounded self-adjoint operator. Prove that $T^n \neq 0$ for any n.

(78) Prove that if a closed subspace M is invariant under a bounded linear operator T (i.e. $Tx \in M$ for all $x \in M$) then $M^\perp$ is invariant under T^*.

(79) Prove that a bounded linear operator T is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in H$.

(80) (HW3) Prove that $KerT = KerT^*T$ and $\overline{RanT^*T} = \overline{RanT^*}$.