FUN Exercise sheet 7. – date of submission 2018.04.19.

(81) (HW1) Let $\mathbf{y} = (y_1, \dots, y_n, \dots)$ be a sequence of complex numbers such that $\sum_{j=1}^{\infty} y_j x_j$ converges for every $\mathbf{x} = (x_1, \dots x_n, \dots) \in c_0$ (recall that c_0 is the space of sequences converging to 0). Prove that $\sum_{n=1}^{\infty} |y_n| < \infty$.

(82) Let x_n be a sequence of vectors in a Banach space, and assume that for all $f \in X'$ the sequence of numbers $f(x_n)$ is bounded. Show that $\|x_n\|$ is bounded.

(83) Let $q_n(t) = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$, and define linear functionals on $C[0, 2\pi]$ by $f_n(\mathbf{x}) = \int_0^{2\pi} \mathbf{x}(t) q_n(t) dt$. Prove that f_n is bounded and $\|f_n\| = \int_0^{2\pi} |q_n(t)| dt$.

(84) Prove that $\int_0^{2\pi} |q_n(t)| dt \to \infty$ as $n \to \infty$.

(85) Let X, Y be Banach spaces. Prove that $X \times Y$ is also a Banach space with the usual norm $\|(x, y)\| = \|x\| + \|y\|$.

(86) Show that an open mapping need not map closed sets to closed sets.

(87) (HW2) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on the space X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are both complete. Assume that for any sequence $(x_n) \subset X$ the fact $\|x_n\|_1 \to 0$ always implies $\|x_n\|_2 \to 0$. Prove that the two norms are equivalent.

(88) (HW3) Let $T : X \to Y$ be an injective bounded linear operator, where X, Y are Banach spaces and $DomT = X$. Prove that $T^{-1} : RanT \to X$ is bounded if and only if $RanT$ is closed in Y.