Practise exercises 1.

(1) Prove by induction that $1^2 + 3^2 + ... + (2n-1)^2 = n(2n-1)(2n+1)/3$.

(2) Prove by induction that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(3) Prove by induction that for every $n > 1$ we have $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

(4) Prove by induction that for all $n > 1$ we have $\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}$

(5) Let $a_0 = 1$ and $a_{n+1} = \sqrt{3a_n + 10}$. Prove that the sequence a_n is monotonically increasing.

(6) Let A, B, C be some sets. Using set operations (intersection, union, complement, etc.) define the following sets:

a; The set of elements of B which are not included in either A or C.

b; The set of elements which belong to exactly two of the sets A, B, C.

c; The set of elements which are not included in all of the three sets.

d; Elements which belong to at most one of the sets.

(7) Write the following statements with logical formulas:

a; There exists an odd natural number larger than 10.

b; Every odd number, which is larger than one, is a prime number.

Write down also the negations of the above statements, both with words and with logical formulas.

(8) Put the following statements into words:

a; $\forall x \in \mathbb{R}((x > 0) \Rightarrow (\exists k \in \mathbb{N}(2^{-k} < x)))$

b; $\exists k \in \mathbb{N}(\forall x \in \mathbb{R}((x > 0) \Rightarrow (2^{-k} < x)))$.

Decide whether the statements are true or false. Write down also the negations of the above statements, both with words and with logical formulas.

(9) Let $P(x)$ mean that x is an even number, and let $H(x)$ mean that x is divisible by 6. Put the following statements into words:

a; $P(4) \wedge H(12)$

b; $\forall x(P(x) \Rightarrow H(x))$

c; $\exists x(P(x) \Rightarrow H(x))$

d; $\exists x(H(x) \Rightarrow \neg P(x))$

Decide whether the statements are true or false. Write down also the negations of the above statements, both with words and with logical formulas.

(10) Write down the equation of a line which:

a; passes through the point (x_0, y_0) and has slope m.

b; passes through the points (x_1, y_1), (x_2, y_2).

Practise exercises 2.

(11) Let $a_n = 1 - \frac{1}{n}$,

$b_n = \frac{2n-1}{n}$,

$c_n = (-1)^n + \frac{1}{n}$,

$d_n = \frac{\ln n}{n^2}$.

Use your calculator to evaluate these expressions for $n = 1000, 10000, 100000$ and $n = 1001, 10001, 100001$. By doing so, get an intuition whether a_n, b_n, c_n, d_n are convergent or not. For the ones that are convergent, provide a threshold index N for $\varepsilon = 0.002$.

(12) Decide whether the following sequences converge and if so, find their limit:

$a_n = \frac{n+3}{4n^2+7n+6}$,

$b_n = \frac{n-5n^4}{n^4+8n^3+1}$,

$c_n = \frac{1-n^3}{70-n^2+n}$.

$d_n = \frac{n^{3/2}+n^2+1}{\sqrt{1+n^2+2\sqrt{n^3+2}}}$.

(13) Decide whether the following sequences converge and if so, find their limit:

$a_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$,

$b_n = \sqrt{n^2-7n+1} - \sqrt{n^2-n+4}$,

$c_n = \sqrt{2n^2+3n+1} - \sqrt{n^2+1}$.

$d_n = (3n+1)(n - \sqrt{n^2+1})$

(14) Find the limit:

$a_n = \sqrt[3]{n^3+3n^2-1} - \sqrt[3]{n^3-2n^2+3n+2}$

(15) Prove the sandwich rule: if $a_n \le b_n \le c_n$ and $a_n \to L$ and $c_n \to L$ then $b_n \to L$.

(16) Prove that if $a_n \to A$ and $b_n \to B$ then $a_n - b_n \to A - B$.

Practise exercises 3.

(17) Prove that if $a < b$ and $0 < h < (b-a)/2$ then $(a+h)(b-h) > ab$.

(18) Prove the order of magnitudes:

$$\frac{(\log n)^\alpha}{n^\beta} \to 0 \text{ for all } \alpha \in \mathbb{R},\ \beta > 0.$$
$$\frac{n^\alpha}{c^n} \to 0 \text{ for all } \alpha \in \mathbb{R},\ c > 1.$$
$$\frac{c^n}{n!} \to 0 \text{ for all } c \in \mathbb{R}.$$
$$\frac{n!}{n^n} \to 0.$$

(19) In class we proved that $\sqrt[n]{c} \to 1$ for all $c > 0$, $\sqrt[n]{n} \to 1$, and $\sqrt[n]{n!} \to +\infty$. Use these facts, the orders of magnitude above, and the sandwich rule to evaluate the following limits:

$$a_n = \sqrt[n]{3n^2}$$
$$a_n = \sqrt[n]{n^3 - n^2 + 4n + 1}$$
$$a_n = \sqrt[n]{4^n + 3n + n^2}$$
$$a_n = \frac{\sin n}{n}$$
$$a_n = \frac{\log(n+1)}{n}$$
$$a_n = \frac{\log n}{\log(2n)}$$
$$a_n = \sqrt[2n+1]{n^2 + \cos n}$$
$$a_n = \frac{\sqrt[4]{n^3+6}}{\sqrt[3]{n^2+3n+2}}$$
$$a_n = \sqrt[n]{1 + \frac{1}{2} + \cdots + \frac{1}{n}}$$

(20) Prove that if a_n is a convergent sequence, it is automatically Cauchy.

(21) Evaluate the following limits:

$$a_n = \left(\frac{n+2}{n}\right)^n$$
$$a_n = \left(\frac{n-1}{n}\right)^n$$
$$a_n = \left(1 + \frac{1}{n^2}\right)^n$$

Practise exercises 4.

(22) Using the formula $\left(1 + \frac{1}{z_n}\right)^{z_n} \to e$ (for $z_n \to +\infty$) evaluate the following limits:

$$a_n = \left(\frac{7n-2}{7n+3}\right)^{3n+1}$$
$$a_n = \left(\frac{n^2+3n-4}{n^2-n+2}\right)^{4n+1}$$
$$a_n = \left(\frac{n^2+\sqrt{n}+1}{n^2-1}\right)^{4n^2+3}$$

(23) Evaluate the sum of the following series:

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)}$

(24) Evaluate the sum of the following series:

$\sum_{n=0}^{\infty} \frac{9}{10^n}$

$\sum_{n=2}^{\infty} \frac{3}{4} \frac{5^{2n}}{3^{3n+1}}$

$\sum_{n=1}^{\infty} \frac{2}{5} \frac{(-3)^{2n+7}}{4^{3n-1}}$

(25) Prove that $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

(26) Using the comparison test and the fact that $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ converges if and only if $\alpha > 1$, decide whether the following series converge or not:

$\sum_{n=1}^{\infty} \frac{n^2-3n+1}{n^3+2n+2}$

$\sum_{n=1}^{\infty} \frac{2n^3+n+7}{n^5-n^2+3}$

$\sum_{n=1}^{\infty} \frac{\sqrt{n+100}}{n+2}$

(27) Using any of techniques learnt (nth term test, ratio test, nth root test, comparison test, absolute convergence, Leibniz criterion, geometric series, partial fractions) decide whether the following series converge or not:

$\sum_{n=1}^{\infty} \frac{7^n (n!)^2}{(2n)!}$

$\sum_{n=1}^{\infty} \frac{n+1}{n^3-1}$

$\sum_{n=1}^{\infty} \frac{10^n}{n!}$

$\sum_{n=1}^{\infty} \frac{\log n}{n}$

$\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n^2})$

$\sum_{n=1}^{\infty} \frac{n \log n}{2^n}$

$\sum_{n=1}^{\infty} \frac{(n!)^2-2^n}{(2n)!}$

$\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$

$\sum_{n=1}^{\infty} \frac{\log n}{n^3}$

$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} (1 + 1/n)^{n^2+1}$

$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

$\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$

$\sum_{n=1}^{\infty} 5(-\frac{1}{3})^{n+3}$ (evaluate!)

$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ (evaluate!)

$\sum_{n=1}^{\infty}(1+1/n)^{\sqrt{n}}$

$\sum_{n=1}^{\infty}\frac{n^2+3n+1}{n^5-7n^3-1}$

$\sum_{n=1}^{\infty}(-1)^n\log(1+1/n)$

$\sum_{n=1}^{\infty}\frac{n^4+n\log n-2^n}{n!+n^{10}+3^n}$

$\sum_{n=1}^{\infty}\frac{n^4}{2^n}$

$\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^{2n^2+1}$

$\sum_{n=1}^{\infty}\frac{\log n+\sqrt{n\log n}}{n^2+1}$

$\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt[n]{n}-1)$

$\sum_{n=1}^{\infty}\frac{2^n n^2}{n!}$

(28) Prove that there exists no real sequence $a_n > 0$ such that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}\frac{1}{a_n}$ both converge.

Practise exercises 5.

(29) For what values of x do the following power series converge?

$\sum_{n=0}^{\infty} x^n$ (evaluate!)

$\sum_{n=1}^{\infty}(x-2)^n$ (evaluate!)

$\sum_{n=0}^{\infty} nx^n$

$\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$\sum_{n=0}^{\infty} n^2x^n$

$\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$

$\sum_{n=1}^{\infty}\frac{1}{n^2}x^n$

$\sum_{n=0}^{\infty}\cos(n\pi/4)\frac{n+1}{n^2+1}x^n$

$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sqrt{n}}(x+3)^n$

(30) Using the product theorem for series evaluate the following sums in closed form:

$\sum_{n=1}^{\infty} nx^n$

$\sum_{n=1}^{\infty} n^2x^n$

$\sum_{n=1}^{\infty}(n+1)^3x^n$

(31) Let $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Using the product theorem for series prove that $f(x+y)=f(x)\cdot f(y)$.

(32) Operations with infinity: prove that if $a_n \to +\infty$ and $b_n \to B \in \mathbb{R}$ then $a_n + b_n \to +\infty$. Also, prove that if $a_n \to +\infty$ and $b_n \to B < 0$ then $a_n b_n \to -\infty$.

(33) In the set of real numbers $\mathbb{R}$ decide whether the subsets $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R} \setminus \mathbb{Q}$ are open, closed, or neither.

(34) Let $H = [-1, 0] \cup \{1/n : n \in \mathbb{N}\}$. Decide whether H is open, closed, or neither.

(35) Prove the fundamental properties of closed sets: $\emptyset, \mathbb{R}$ are closed, the finite union of closed sets is closed, and the intersection of arbitrarily many closed sets is closed.

(36) Prove that for any real values $a_1, \dots, a_n$ the finite set $\{a_1, \dots, a_n\} \subset \mathbb{R}$ is closed.

(37) Prove that if $G \subset \mathbb{R}$ is an open set then its shifted version $x + G = \{x + y : y \in G\}$ is also open for any $x \in \mathbb{R}$.

(38) Prove that if G is open, F is closed then $G \setminus F$ is open, and $F \setminus G$ is closed.

(39) Give an example of infinitely many closed sets such that their union is open.

Practise exercises 6.

(40) Find the following limits:

$\lim_{x \to +\infty} \frac{2x^2+3x-1}{3x^2+5x-4}$

$\lim_{x \to +\infty} \frac{4x^3+x+3}{3x^4+6x+4}$

$\lim_{x \to +\infty} \frac{x^{3/2}+x+\log x}{\sqrt{x^3+x+1}}$

$\lim_{x \to -1} \frac{x+3}{x^2+3x+1}$

$\lim_{x \to 1} \frac{x^2-1}{x-1}$

$\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$

$\lim_{y \to 1} \frac{y^2-3y+2}{y^2-1}$

$\lim_{x \to 5_+} \frac{x+3}{x-5}$

$\lim_{x \to 5_-} \frac{x+3}{x-5}$

(41) Prove that $\lim_{x \to 0} \cos(1/x)$ does not exist.

(42) Prove that $\lim_{x \to 0} x \cos(1/x) = 0$.

(43) Let $f(x) = \frac{x^2+x-2}{x^2-1}$ if $x \neq 1$, and $f(1) = 4$. Is f continuous at the point $x_0 = 1$?

(44) Let $f(x) = \frac{x^2+x+a}{x-3}$ if $x \neq 3$, and $f(3) = b$. Choose the parameters a and b so that the function f be continuous at $x_0 = 3$.

(45) Let $f(x) = \frac{x^2+1}{x^2} - \frac{1}{\cos x}$. Prove that f has a zero in the open interval $(0, \pi/2)$ (use limits at the endpoints, and the intermediate value theorem).

(46) Prove that if f and g are continuous at $x_0 \in \mathbb{R}$ then $f+g$ is also continuous at x_0.

(47) Use the algebraic rules and the chain rule to find the derivative of the following functions:

$$f(x) = \cos(x^3 + 3x - 1)$$
$$f(x) = (x^3 + 3x)(\sin x + \cos x)$$
$$f(x) = (x^2 + 1)^{17}$$
$$f(x) = \cos(2x)\sin(x^2 - 1)$$
$$f(x) = \sqrt{x} \text{ (use the definition of derivative here!)}$$
$$f(x) = 1/x^3$$
$$f(x) = (4x^3 + x - 3)\cos(2x + 1)\sin(x^2 - 7x + 1)$$

(48) In class we proved $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Use this to evaluate the following limits:

$$\lim_{x\to 0} \frac{\tan 3x}{\sin 7x}$$
$$\lim_{x\to 0} \frac{\sin 3x}{5x}$$
$$\lim_{x\to 0} \sin 5x \cot 4x$$

(49) In class we proved that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$. Follow the method of the proof to conclude that $\lim_{x\to 0} \frac{1-\cos x}{x^2} = 1/2$.

(50) Use the result of the previous exercise to evaluate the following limits:

$$\lim_{x\to 0} \frac{-1+\cos 3x}{7x^2}$$
$$\lim_{x\to 0} \frac{1-\cos 5x}{x \sin 3x}$$
$$\lim_{x\to 0} \frac{1-\cos 4x}{\tan 3x \sin 7x}$$

(51) Differentiate the following functions:

$$f(x) = \tan x$$
$$f(x) = \cos(\sqrt[3]{x^3 + \sqrt{x} + 1})$$
$$f(x) = \tan(x^2 + 1)\sin(1/x)$$
$$f(x) = \cot\left(\frac{x^2+3}{\sqrt{\sin(2x-1)+7}}\right)$$
$$f(x) = \tan 3x \cos 5x \sin 7x$$

The first midterm test may include problems up to this point.

Practise exercises 7.

(52) Use linear approximation (choosing your starting point x_0 appropriately) to approximate the following values:

$$\sqrt{65},\ \sqrt[3]{65}, \cos(\tfrac{\pi}{9}), \tan(\tfrac{\pi}{5}).$$

(53) Use the definition of the derivative to prove that for $f(x) = \sqrt[3]{x}$ we have $f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$

(54) Use your experience with the previous exercise to prove that for any positive integer q and $f(x) = \sqrt[q]{x}$ we have $f'(x) = \frac{1}{q}x^{\frac{-q+1}{q}}$.

(55) Use the chain rule and the previous exercise to prove that for any positive integers p, q and $f(x) = x^{p/q}$ we have $f'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$

(56) Having worked through the previous three exercises, now use implicit differentiation of $y^q = x^p$ to give a simple proof of $\frac{dy}{dx} = \frac{p}{q}x^{\frac{p}{q}-1}$.

(57) Let $2xy + \pi \sin y = 2\pi$. Give $\frac{dy}{dx}$ at the point $(1, \pi/2)$.

(58) Find the equation of the tangent line to the curve $x \sin 2y = y \cos 2x$ at the point $(\pi/4, \pi/2)$.

(59) Find the first and second derivatives $(\frac{dy}{dx}, \frac{d^y}{dx^2})$ at the point $(1, 1)$ of the curve $y^2(2 - x) = x^3$.

(60) Differentiate the following functions:

$$f(x) = e^{3x^4+x+1}\log(x^2 + 1)$$

$$f(x) = \frac{\arctan(3x^2+4)\cos(\sqrt{2x+3})}{\log(\sin 3x)}$$

$$f(x) = \arccos(x^3 - x + 1)e^{\sin(\sqrt{x^2+3})}$$

$$f(x) = \arcsin(1 - e^{3x}) + \arctan(2^x + 1)$$

$$f(x) = x^{\sqrt{x}}$$

$$f(x) = x^{x \log x}$$

Practise exercises 8.

(61) Use L'Hospital's rule to evaluate the following limits:

$\lim_{x\to 0} \frac{\log(1+x)}{x}$

$\lim_{x\to\pi/2} \frac{2x-\pi}{\cos x}$

$\lim_{x\to 0} \frac{x-\sin x}{x^3}$

$\lim_{x\to 0} \frac{3^{\sin x}-1}{x}$

$\lim_{x\to\pi/2}(\frac{\pi}{2}-x)\tan x$

$\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$

$\lim_{x\to 0} \frac{x(1-\cos x)}{x-\sin x}$

$\lim_{x\to 0_+} x^x$

$\lim_{x\to 1_+} x^{1/(x-1)}$

$\lim_{x\to 0}(e^x + x)^{1/x}$

(62) By comparing the derivatives and initial values of the two functions in question, prove the following inequalities:

$\log(1+x) \leq x$ for all $x \geq 0$

$e^{x^2} \geq 1 + x^2$ for all $x \geq 0$

$\cos x \geq 1 - \frac{x^2}{2}$ for all $x \in \mathbb{R}$

$\sin x \geq x - \frac{x^3}{6}$ for all $x \geq 0$

$\arctan x \leq x$ for all $x \geq 0$

Practise exercises 9.

(63) Analyze the following functions and sketch their graphs:

$y = x^3 - 3x + 3$

$y = 2x^4 - 4x^2 + 1$

$y = x + \sin x,\ x \in [0, , 2\pi]$

$y = \frac{x^2}{x-1}$

$y = \frac{x}{x^2-1}$

$y = \frac{4x}{x^2+4}$

$y = \frac{8}{x^2+4}$

$y = (x-3)e^{-x}$

(64) What is the maximal area of a rectangle inscribed in a semi-circle of radius 1?

(65) Assume you want to make a cylindrical can such that the top and the bottom are reinforced by using two layers of tin. What is the minimal material of tin you need to use to make such a can of volume 1 liter?

(66) Your company can sell x items per week for a revenue of $r(x) = 200x - 0.00x^2$ cents. The cost of making x items per week is $c(x) = 50x + 20000$ cents. What is your maximal profit?

(67) A rectangular plot of farmland will be bounded on one side by a river and on the three other sides by electric fence. You have 800 meters of fence available. What is the largest area that you can enclose?

(68) Find the maximal volume of a cylinder inscribed in a sphere of radius 1.

(69) You want to make a cylindrical tin cup (open top!) of volume 1 liter. What is the minimal possible surface area of the cup?

(70) Let $f(x) = x^2(2 + \sin(1/x))$ for $x \neq 0$, and $f(0) = 0$. Prove that f has a local minimum at 0, but there is no $\varepsilon > 0$ such that f' is negative on $(-\varepsilon, 0)$, and positive on $(\varepsilon, 0)$.

(71) Find the following Taylor polynomials of given order and center:

$f(x) = \sin x$, center $x_0 = \pi/3$, order 3,

$f(x) = 2^x$, center $x_0 = 1$, order 3,

$f(x) = \tan x$, center $x_0 = 0$, order 3,

$f(x) = \tan x$, center $x_0 = \pi/4$, order 2,

$f(x) = \log(1 - x)$, center $x_0 = 1$, order 4,

$f(x) = \arccos x$, center $x_0 = 0$, order 2.

(72) Let $P(x)$ be a polynomial of degree n. Prove that the Taylor polynomial of order n corresponding to $P(x)$ at any center $x_0 \in \mathbb{R}$ is $P(x)$ itself.

(73) Estimate the value of $\sqrt{65}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt{x}$, at center 64. Give an upper bound on the error of approximation.

(74) Estimate the value of $\log 1.2$ by the Taylor polynomial of order 3 of $f(x) = \log(1+x)$, at center 0. Give an upper bound on the error of approximation.

(75) Calculate the Taylor series of the following functions, with given center:

$f(x) = \sin 2x$, center $x_0 = 0$,

$f(x) = e^x$, center $x_0 = 1$,

$f(x) = \frac{1}{1+x^3}$, center $x_0 = 0$,

$f(x) = \frac{x^2}{1+x^3}$, center $x_0 = 0$,

$f(x) = \frac{1}{(1-x)^2}$, center $x_0 = 0$.

(76) Find the following basic integrals:

$\int 4x^7 dx$

$\int \frac{5}{x^2} dx$

$\int 3\sqrt{x} dx$

$\int \sin 3x dx$

$\int \frac{3}{1+x^2} dx$

(77) Find the following integrals by the method of integration by parts:

$\int x^2 e^x dx$

$\int x^2 \cos x dx$

$\int x \ln x dx$

$\int e^x \cos x dx$

$\int x \arctan x dx$

$\int \arctan x dx$

$\int x e^{-x} dx$

$\int x^2 \sin 2x dx$

(78) Solve the following initial value problems (give y as a function of x):

$\frac{dy}{dx} = x^2 + 1$, $y = 1$ when $x = 0$

$\frac{dy}{dx} = x^2 + \sqrt{x}$, $y = 1$ when $x = 1$

$\frac{dy}{dx} = -5/x^2$, $x > 0$, $y = 3$ when $x = 5$

$\frac{dy}{dx} = 1 + \cos x$, $y = 4$ when $x = 0$.

(79) In a cylindrical water tank the height of the water is $y_0 = 9$ meters. We open a tap at the bottom of the tank, and the water starts flowing out. When the water level stands at y in the tank, the rate of decrease of the water level is given by $\frac{dy}{dt} = -0.1\sqrt{y}$ (where t is measured in minutes and y in meters). Find the water level y as a function of time. How long does it take the tank to drain?

Practise exercises 11.

(80) Find the following integrals by the method of substitution (exploit that $g'(x)dx$ appears in the integrals):

$\int \sin^7 x \cos x dx$

$\int \frac{2x+1}{\sqrt[4]{x^2+x+3}} dx$

$\int 3x e^{2x^2+1} dx$

$\int \frac{-x}{\sqrt{4-x^2}} dx$

$\int x \sin(2x^2) dx$

$\int \frac{9r^2}{\sqrt{1-r^3}} dx$

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

$\int \frac{\cos x}{\sqrt{2+\sin x}}dx$

$\int \sin^6 x \cos^3 x dx$

$\int \frac{1}{x\ln x}dx$

(81) Find the following integrals by the method of substitution ($u = ax + b$ or $x = \sin t$:

$\int \cos(3x-4)dx$

$\int (4x+1)^7 dx$

$\int \frac{1}{1+(2x-1)^2}dx$

$\int \frac{1}{4+(7x+3)^2}dx$

$\int \frac{1}{4x^2+4x+2}dx$

$\int \frac{x^2}{\sqrt{1-x^2}}dx$

$\int \frac{1}{4-x^2}dx$

$\int \frac{1}{\sqrt{2x-x^2}}dx$

$\int \frac{4x^2}{(1-x^2)^{3/2}}dx$

Practise exercises 12.

(82) Find the integral of the following rational functions:

$\int \frac{1}{1-x^2}dx$

$\int \frac{1}{x^2+2x}dx$

$\int \frac{x}{x^2-2x-3}dx$

$\int \frac{x^2}{(x-1)(x+1)^2}dx$

$\int \frac{x^3}{x^2+1}dx$

$\int \frac{3x^2+x+4}{x^3+x}dx$

$\int \frac{x}{x^2+x+1}dx$

$\int \frac{4x+4}{x^2(x^2+1)}dx$

(83) Use the substitution $z = \tan(x/2)$ to find the following integrals:

$\int \frac{1}{1+\sin x}dx$

$\int \frac{1}{1-\cos x}dx$

$\int \frac{\cos x}{1-\cos x}dx$

$\int \frac{1}{\sin x-\cos x}dx$

(84) Use the substitution $t = e^x$ to find the following integrals:

$\int \frac{e^x}{1-e^{2x}} dx$

$\int \frac{1}{1+e^x} dx$

$\int \frac{1}{1-e^{3x}} dx$

(85) Use the substitution $t = \sqrt{ax+b}$ to find the following integrals:

$\int \frac{x}{\sqrt{x+1}} dx$

$\int \frac{\sqrt{2x+1}}{x+3} dx$

(86) Use the substitution $x = \sinh t$ or $x = \cosh t$ to find the following integrals:

$\int \sqrt{x^2-1} dx$

$\int \sqrt{x^2+1} dx$

$\int \sqrt{4x^2-2} dx$

(87) Find the integral of the following inverse functions (use integration by parts):

$\int \arcsin x dx$

$\int \arccos x dx$

$\int \arctan x dx$

Practise exercises 13.

(88) Find the value of the following definite integrals:

$\int_0^3 8x^3 dx$

$\int_0^\pi (1+\cos x) dx$

$\int_{-1}^1 \sqrt{1-x^2} dx$

$\int_{-1}^1 \sqrt{1+x^2} dx$ (Hint: use $x = \sinh t$)

$\int_0^1 \frac{1}{1+x^2} dx$

(89) Decide whether the following improper integrals converge ($< +\infty$) or diverge ($= +\infty$). For those that converge, find their value.

$\int_0^1 \cot x dx$

$\int_5^\infty \frac{1}{x^2-4} dx$

$\int_0^{\pi/2} \frac{1}{\sin x} dx$

$\int_3^\infty \frac{x^{3/2}+1-\sqrt{x}}{x^2+1} dx$

$\int_0^1 \sqrt{\frac{1+x}{1-x}} dx$

(90) Prove that $\int_1^\infty \frac{1}{x(\log x)^\alpha} dx$ is finite if and only if $\alpha > 1$.

(91) Find the arc-length of the following curves on the given intervals:

$y = \frac{1}{3}(x^2+2)^{3/2}$, $x \in [0,3]$

$y = \cosh x$, $x \in [-\log 2, \log 2]$

$y = x^2$, $x \in [0,1]$

(92) Find the volume of the following bodies of rotation (we always rotate the given curve around the x-axis over the given interval):

$y = \sqrt{x}$, $x \in [0,4]$

$y = e^x$, $x \in [0,2]$

$y = \sqrt{\cos x}$, $x \in [0,\pi/2]$

$y = \frac{1}{\cos x}$, $x \in [0,\pi/4]$

(93) Find the surface of the following bodies of rotation (we always rotate the given curve around the x-axis over the given interval):

$y = x^3$, $x \in [0,2]$

$y = \sqrt{x+1}$, $x \in [1,5]$

The second midterm test may include problems up to this point.
