TB, SD, UK, EJP/281959, 23/08/2008

IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **29** (2008) 1–5 UNCORRECTED PROOF

Coordinate time and proper time in the GPS

T Matolcsi[1,4] **and M Matolcsi**[2,3,5]

[1] Department of Applied Analysis, Eötvös University, Pázmány P. sétány 1C., H–1117 Budapest, Hungary
[2] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13–15, H–1053, Budapest, Hungary
[3] BME, Department of Analysis, Egry J. u. 1, Budapest, Hungary

E-mail: matolcsi@cs.elte.hu and matomate@renyi.hu

Received 22 May 2008, in final form 6 August 2008
Published DD MMM 2008
Online at stacks.iop.org/EJP/29/1

Abstract

The global positioning system (GPS) provides an excellent educational example as to how the theory of general relativity is put into practice and becomes part of our everyday life. This paper gives a short and instructive derivation of an important formula used in the GPS, and is aimed at graduate students and general physicists. The theoretical background of the GPS (see [1]) uses the Schwarzschild spacetime to deduce the *approximate* formula, $\mathrm{d}s/\mathrm{d}t \approx 1 + V - \frac{|\mathbf{v}|^2}{2}$, for the relation between the proper time rate s of a satellite clock and the coordinate time rate t. Here V is the gravitational potential at the position of the satellite and $\mathbf{v}$ is its velocity (with light-speed being normalized as $c = 1$). In this paper we give a different derivation of this formula, *without using approximations*, to arrive at $\mathrm{d}s/\mathrm{d}t = \sqrt{1 + 2V - |\mathbf{v}|^2 - \frac{2V}{1+2V}(\mathbf{n} \cdot \mathbf{v})^2}$, where $\mathbf{n}$ is the normal vector pointing outwards from the centre of Earth to the satellite. In particular, if the satellite moves along a circular orbit then the formula simplifies to $\mathrm{d}s/\mathrm{d}t = \sqrt{1 + 2V - |\mathbf{v}|^2}$. We emphasize that this derivation is useful mainly for educational purposes, as the approximation above is already satisfactory in practice.

1. Introduction

The most significant application of the theory of general relativity in everyday life, arguably, is the global positioning system. The GPS uses accurate, stable atomic clocks in satellites and on the ground to provide worldwide position and time determination. These clocks have relativistic frequency shifts which need to be carefully accounted for, in order to achieve

[4] Supported by OTKA-T048489.
[5] Supported by OTKA-PF 64061, T 049301, T047276.

synchronization in an underlying Earth-centred inertial frame, upon which the whole system is based. For educational purposes it is very instructive to see how a highly abstract theory such as the general theory of relativity becomes a part of our everyday life.

Throughout this paper we will normalize universal constants for simplicity, so that light speed and the gravitational constant are the unity, $c = 1, \gamma = 1$.

Let us briefly sketch here how the GPS works, including the major ingredients for the reader's convenience (a full and detailed description is available in [1]). In order to determine your position at the surface of Earth you must be able to see some (at least four) GPS satellites simultaneously, and know their position and your distance to them (with this much information at hand, it is then elementary geometry to determine your position). To make this possible, every GPS satellite emits signals continuously, describing its position and its local time t_{sat}. If your GPS device measures its own local time, t_{dev}, then you can use the constancy of the speed of light c to calculate your distance d to the satellite as

$$d = (t_{\text{dev}} - t_{\text{sat}})c = t_{\text{dev}} - t_{\text{sat}} \tag{1}$$

keeping in mind the normalization $c = 1$. (We remark here that in practice the GPS device is unable to measure time with sufficient accuracy, therefore t_{dev} also has to be deduced from the data sent by the satellites. However, this detail is not important for the purposes of this paper.)

This looks simple enough, but the problem is that formula (1) is only valid if all clocks are *synchronized* in some special relativistic *inertial frame*. Therefore, for the purposes of the GPS one imagines that an inertial frame is attached to the centre of Earth, and we try to synchronize all clocks such that they measure the time t of this ideal inertial frame. This means that one should imagine 'ideal' clocks placed everywhere in the vicinity of Earth, measuring the time t, and thus t_{sat} and t_{dev} should be the read-outs of these ideal clocks at the place of the satellite when the signal originates and at the place of your device when the signal is received. However, what we *actually can* measure is the proper time t_E of stationary clocks on the surface of Earth (which is measured in time-keeping centres throughout the world), and the proper time s of satellite clocks. Therefore, we need to establish a relation between the time rates t_E, s and t.

Fortunately, the time rate t_E measured on Earth is independent of where you are (i.e. the same manufactured clocks beat the same rate in London or Tokyo or New York) and differs from the 'ideal' time rate t only by a multiplicative constant,

$$\frac{\mathrm{d}\,t}{\mathrm{d}\,t_E} = 1 - \Phi_0, \tag{2}$$

where Φ_0 is a constant corresponding to the Earth's geoid (see [1]). This relation is very convenient because the ideal time rate t can be replaced by t_E, something we can actually measure, and an equation of form (1) still remains valid after rescaling by the factor $1 - \Phi_0$. This leaves us with the task of establishing a relation between s and t (or s and t_E, whichever turns out to be more convenient).

To determine the relation of the proper time rate s of the satellite clocks and the time rate t of ideal clocks measuring the coordinate time of the underlying Earth-centred inertial frame, the customary theoretical framework [1] is to use Schwarzschild spacetime, and arrive at the formula

$$\mathrm{d}\,s/\mathrm{d}\,t \approx 1 + V - \frac{|\mathbf{v}|^2}{2}, \tag{3}$$

after several first-order approximations in the calculations. Here V denotes the gravitational potential at the position of the satellite and $\mathbf{v}$ is the velocity of the satellite measured in the underlying non-rotating Earth-centred inertial frame. Formula (3) is the internationally accepted standard relating the clock frequencies, as described in [1, 3] and references therein.

We remark here that the derivation of formula (2) is somewhat more involved than that of formula (3). In fact, for the purposes of deriving (2) one needs to *modify* the Schwarzschild metric by a small term, taking into account multipole contributions corresponding to the Earth's geoid. However, for formula (3) the modifying term is *disregarded* and the standard Schwarzschild metric is used, the argument being that the modifying term becomes negligible far enough from the Earth surface, where the satellites orbit (cf [1]). In this paper we do not include the derivation of formula (2) and the modifying term which is used there.

In general, in physics it is justified to use approximate formulae for two different reasons. The one is that in some cases the derivation of an exact result is analytically not possible. The other is that in certain cases an exact analytic derivation, even if it exists, would lead to involved and lengthy calculations thus concealing the important and possibly very simple aspects at the heart of the issue. In calculations involving the theory of relativity (in particular, general relativity) there is a tendency to turn to approximations *automatically*, due to the involved nature of the theory. However, in some rare cases a modified point of view and an adequate choice of the coordinate system can lead to exact results.

In this paper, *we adhere to the standard theoretical framework* used in [1, 3], but we point out that an *exact formula can be derived* in a very simple manner for the clock frequency rate (3) in question. Instead of using the customary isotropic coordinates we will treat Schwarzschild spacetime with Schwarzschild coordinates (an entirely coordinate-free derivation of the same formulae can be found in [5] but it is rather cumbersome). This treatment of Schwarzschild spacetime is motivated by a similar account of special relativity in [4] and that of general relativity in [7]. The coordinate-free point of view often has the advantage of conceptual clarity and, in this particular case, brevity of calculations.

Our results here are mostly of educational and theoretical interest as the existing formula (3) provides good approximation to the desired precision in the GPS (see [3]).

2. Schwarzschild's spacetime

Schwarzschild's spacetime describes the gravitational field of a pointlike inertial mass m. It is a well-known model of general relativity, but we include its short description here for convenience.

Let us introduce some notation. Let E be a three-dimensional Euclidean vector space, the inner product of $\mathbf{x}, \mathbf{y} \in E$ being denoted by $\mathbf{x} \cdot \mathbf{y}$. For $0 \neq \mathbf{x} \in E$ we put

$$\mathbf{n}(\mathbf{x}) := \frac{\mathbf{x}}{|\mathbf{x}|} \tag{4}$$

for the outward normal vector at $\mathbf{x}$.

Consider $\mathbb{R} \times E$ as a spacetime manifold with its usual special relativistic metric, i.e. the Lorentz form given by

$$\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}, \tag{5}$$

where I is the identity matrix. For a world point $(t, \mathbf{x}) \in \mathbb{R} \times E$ one should think of t as the synchronization time corresponding to the centre of Earth, and $\mathbf{x}$ as the space vector pointing to the world point from the centre of Earth. The Lorentz form (5) means that the Lorentz length square of a 4-vector $(s, \mathbf{q})$ is given by the usual formula $|(s, \mathbf{q})|^2_{\mathrm{Lor}} = -s^2 + |\mathbf{q}|^2$. This special relativistic spacetime will be called the Earth-centred reference frame (ECRF). This is an 'ideal' special relativistic frame and the task in the GPS is to achieve synchronization of clocks in this underlying inertial frame.

Introduce the potential $V(\mathbf{x}) := -\frac{m}{|\mathbf{x}|}$ on E, and restrict your consideration to world points $(t, \mathbf{x}) \in \mathbb{R} \times E$ for which $1 + 2V(\mathbf{x}) > 0$ (i.e. for world points outside the Schwarzschild radius). Then it is easy to see (cf [2, 6, 8]) that for such world points, the standard form of the Schwarzschild metric in $\mathbb{R} \times E$ (i.e. a smooth collection of Lorentz forms $g(t, \mathbf{x})$ depending on the world points) takes the form:

$$g(t, \mathbf{x}) = \begin{pmatrix} -(1 + 2V(\mathbf{x})) & 0 \\ 0 & I - \frac{2V(\mathbf{x})}{1+2V(\mathbf{x})} \mathbf{n}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x}) \end{pmatrix}. \tag{6}$$

In particular, when we measure the length square of any *space vector* $\mathbf{q}$ at the space point $\mathbf{x}$ in Schwarzschild's metric we obtain

$$|\mathbf{q}|^2_{\text{Sch}} := |\mathbf{q}|^2 - \frac{2V(\mathbf{x})}{1 + 2V(\mathbf{x})} (\mathbf{n}(\mathbf{x}) \cdot \mathbf{q}))^2. \tag{7}$$

Similarly, when we measure the Schwarzschild length square of a *4-vector* (e.g. a 4-velocity) $(s, \mathbf{q})$ at the point $\mathbf{x}$ we get

$$|(s, \mathbf{q})|^2_{\text{Sch}} = -(1 + 2V(\mathbf{x}))s^2 + |\mathbf{q}|^2_{\text{Sch}}. \tag{8}$$

3. A satellite in the Earth's gravitational field

In this section we derive the formula relating the proper time s of satellite vehicle clocks to the ideal time t of the underlying inertial frame.

A material point in spacetime is described by a world line function

$$\mathbb{R} \to \mathbb{R} \times E, \qquad s \mapsto (t(s), x(s)), \tag{9}$$

where s is the proper time of the material point. The 4-velocity, $(\dot{t}(s), \dot{x}(s))$, of the material point always satisfies

$$|(\dot{t}(s), \dot{x}(s))|^2_{\text{Sch}} = -1. \tag{10}$$

Therefore, using equation (8) we obtain

$$-(1 + 2V(x(s)))\, \dot{t}(s)^2 + |\dot{x}(s)|^2_{\text{Sch}} = -1 \tag{11}$$

The time function $t(s)$ can be inverted, with $s(t)$ denoting the proper time instant s corresponding to the ECRF-time t. Let $\mathbf{v}(t) := \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ denote the relative velocity of the material point with respect to the centre of the Earth. Note that by the chain rule we have $\mathbf{v}(t) = \dot{x}(s(t))\frac{\mathrm{d}s}{\mathrm{d}t}$. Now, rearranging (11) we obtain

$$\frac{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + |\dot{x}(s(t))|^2_{\text{Sch}} \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{1 + 2V(x(s(t)))} = 1, \tag{12}$$

which yields

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1 + 2V(x(s(t))) - |\mathbf{v}(t)|^2_{\text{Sch}}}. \tag{13}$$

Finally, applying (7) we obtain the desired relation

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1 + 2V - |\mathbf{v}|^2 - \frac{2V}{1 + 2V} (\mathbf{n} \cdot \mathbf{v})^2}. \tag{14}$$

Using a series expansion, assuming that all the terms on the right-hand side are much less than 1, we get back the *approximate* formula (3)

$$\frac{\mathrm{d}s}{\mathrm{d}t} \approx 1 + V - \frac{|\mathbf{v}|^2}{2}. \tag{15}$$

Of course, there is still some work to be done before one can apply equation (1) in the GPS. Instead of $\frac{ds}{dt}$ what we really need is the function $t(s)$, because the signal emitted by the satellite contains the proper time instant s measured by clocks on board, and we would like to replace it by the coordinate-time instant t which it corresponds to. However, having deduced formula (14) (or its approximation (3)) it is possible to obtain the function $t(s)$, or at least a good enough approximation of it. This is described very well in full detail in [1], and we feel it inappropriate to repeat those calculations word-by-word here. Nevertheless, we warmly recommend that the reader turn to [1] for the interesting details.

References

[1] Ashby N 2005 Relativity in the global positioning system *100 Years of Relativity* (Hackensack, NJ: World Scientific) pp 257–89 (also at Livingreviews in Relativity, http://relativity.livingreviews.org/Articles/lrr-2003-1/)

[2] Bergmann P G 1978 *Introduction to the Theory of Relativity* (New York: Dover) chapter 13, p 203, equation (13.25)

[3] Kouba J 2004 Improved relativistic transformations in GPS *GPS Solutions* **8** 170–80

[4] Matolcsi T 1993 *Spacetime Without Reference Frames* (Budapest: Akadémiai Kiadó)

[5] Matolcsi T and Matolcsi M 2006 GPS revisited: the relation of proper time and coordinate time arXiv:math-ph/0611086v1

[6] Pauli W 1981 *Theory of Relativity* (New York: Dover) p 166, equation (421*a*)

[7] Sachs R K and Wu H 1977 *General Relativity for Mathematicians* (New York: Springer)

[8] Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley) p 181

Endnotes

(1) Author: Please check the authors affiliations and footnotes as set.
(2) Author: There is some discrepancy between manuscript and source file. Hence the source file is followed as it seems to be the updated one.
