Thomas rotation and Thomas precession

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Abstract

Exact and simple calculation of Thomas rotation and Thomas precessions along a circular world line is presented in an absolute (coordinate-free) formulation of special relativity. A straightforward derivation of the Fermi-Walker equation is also given. Besides the simplicity of calculations the absolute treatment of spacetime allows us to make a clear conceptual distinction between the phenomena of Thomas rotation and Thomas precession.

Key words: Thomas rotation, Thomas precession, gyroscope

1 Introduction

The 'paradoxic' phenomenon of Thomas precession has given rise to much discussion ever since the publication of Thomas' seminal paper (Thomas 1927) in which he made a correction by a factor 1/2 to the angular velocity of the spin of an electron moving in a magnetic field. Let us mention here that in the literature there seems to be no standard agreement as to the usage of the terms 'Thomas precession' and 'Thomas rotation'. As explained in more detail in Section 10 below, we prefer to use the term *Thomas precession* to refer to the continuous change of direction, *with respect to an inertial frame*, of a gyroscopic vector moving along a world line. *Thomas rotation*, on the other hand, will refer to the spatial rotation experienced by a gyroscopic vector having moved along a 'closed' world line, and having returned to its initial frame of reference (see Section 9).

One of the most studied cases (see e.g. Costella et al. 2001, Kennedy 2002) is the fact that the application of three successive Lorentz boosts (with the relative velocities adding up to zero) results, in general, in a spatial rotation: the *discrete* Thomas rotation (see Section 4 for details). The same fact is often described as 'the composition of two Lorentz boosts is equivalent to a boost and a spatial rotation'. We prefer to use three Lorentz boosts instead (with the relative velocities adding up to zero), in order to return to the initial frame of reference, in accordance with our terminology of Thomas rotation. Describing the mathematical structure of discrete Thomas rotations has motivated A.A. Ungar to build the comprehensive theory of gyrogroups and gyrovector spaces (Ungar 2001).

The other case typically under consideration comes from the original observation of Thomas: the continuous change of direction, with respect to an inertial frame, of a gyroscopic vector moving along a circular orbit. This phenomenon has been subject to considerations from various points of view (Muller 1992 (Appendix), Philpott 1996, Rebilas 2002 (Appendix), Herrera & Di Prisco 2002, Rhodes & Shemon 2003). The considerations usually involve, either explicitly or implicitly, the viewpoint of the orbiting 'airplane', i.e. a rotating observer. This might lead us to believe (see Herrera & Di Prisco 2002) that the calculated angle of rotation depends on the definition of the rotating observer (and this could lead to an experimental checking of what the 'right' definition of a rotating observer is). From our treatment below, however, it will be clear the Thomas rotation is an absolute fact, independent of the rotating (or, any other) observer.

It is also interesting to note that new connections between quantum mechanical phenomena and Thomas rotation have recently been pointed out (Lévay 2004).

As it is well known, the theory of special relativity contradicts our common sense notions about space and time in many respects. Early day 'paradoxes' were usually based on our intuitive assumption of absolute simultaneity. With the resolution of paradoxes such as the 'twin paradox' or the 'tunnel paradox' it has become common knowledge that the concept of time must be handled very carefully. As it is also well known, the theory of special relativity implies, besides the non-existence of absolute time, the *non-existence of absolute space*. An expression such as 'a point in space' simply does not have an absolute meaning, just as the expression 'an instant in time'. However, this fact seems to be given less attention to and even overlooked sometimes. The fact that the space vectors of any observer are usually represented as vectors in $\mathbb{R}^3$ leads one to forget that these spaces really are *different*. This conceptual error lead e.g. to the 'velocity addition paradox' (Mocanu 1992). The spaces of two different inertial observers are, of course, connected via the corresponding Lorentz boost, and the non-transitivity of Lorentz boosts (which, in fact, gives rise to the notion of Thomas rotation) gave the correct explanation of this 'paradox' (Ungar 1989, Matolcsi & Goher 2001).

To grasp the essence of the concepts related to Thomas rotation, let us mention that in some sense this intriguing phenomenon is analogous to the well known twin paradox. Consider two twins in an inertial frame. One of them remains in that frame for all times, while the other goes for a trip in spacetime, and later returns to his brother. It is well-known that different times have passed for the two twins: the traveller is younger than his brother. What may be surprising is that the space of the traveller when he arrives, although he experienced no torque during his journey, will be rotated compared to the space of his brother; this is, in fact, the Thomas rotation. This analogy is illuminating in one more respect: until the traveller returns to the original frame of reference it makes no sense to ask 'how much younger is the traveller compared to his brother?' and 'by what angle is the traveller's gyroscope rotated compared to that of his brother?' Different observers may give different answers. When the traveller returns to his brother, these questions suddenly make perfect sense, and there is an absolute answer (independent of who the observer is) as to how much younger and how much rotated the traveller is.

Of course, an arbitrtary inertial frame can observe the brothers continuously, and can tell, at each of the frame's instants, what difference *he sees* between the ages of the brothers. More explicitly, as it is well known, given a world line, an arbitrary inertial frame can tell the relation between the frame's time and the proper time of the world line. This relation depends on the inertial frame: different inertial frames establish different relations.

Similarly, an arbitrary inertial frame, observing the two brothers, can tell at each frame-instant what difference *he sees* between the directions of the gyroscopes of the brothers. Different inertial frames establish different relations.

This philsophy makes a clear distinction between Thomas rotation and Thomas precession connected to a world line:

– Thomas rotation refers to an absolute fact (independent of who observes it), which makes sense only for two equal local rest frames (if such exist) of the world line,

– Thomas precession refers to a relative fact (i.e. depending on who observes the motion), which makes sense with respect to an arbitrary inertial frame.

In this paper we use the formalism of (Matolcsi 1993) to give a concise and rigorous treatment of the discrete and circular-path Thomas rotations. The Thomas rotation as well as the Thomas precession (with respect to certain inertial observers) along a circular world line are calculated. Our basic concept here is that special relativistic spacetime has a four-dimensional affine structure, and coordinatization (relative to some observer) is, in many cases, unnecesary in the description of physical phenomena. In fact, coordinates can sometimes lead to ambiguities in concepts and definitions, and bear the danger of leading us to overlook the fact that absolute space does not exist.

As well as providing a clear overview of the appearing concepts, the coordinate-free formulation of special relativity enables us to give simple calculations. The indispensable Fermi-Walker equation is also straightforward to derive in our formalism.

2 Fundamental notions

In this section some notions and results of the special relativistic spacetime model as a mathematical structure (Matolcsi 1993, 1998, and Matolcsi & Goher 2001) will be recapitulated. As the formalism slightly differs from the usual textbook treatments of special relativity (but only the formalism: our treatment is mathematically equivalent to the usual treatments), we will point out several relations between textbook formulae and those of our formalism.

Special relativistic spacetime is an oriented four dimensional affine space M over the vector space $\mathbf{M}$; the spacetime distances form an oriented one dimensional vector space $\mathbf{I}$, and an arrow oriented Lorentz form $\mathbf{M} \times \mathbf{M} \to \mathbf{I} \otimes \mathbf{I}$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$ is given.

An *absolute velocity* $\mathbf{u}$ is a future directed element of $\frac{\mathbf{M}}{\mathbf{I}}$ for which $\mathbf{u} \cdot \mathbf{u} = -1$ holds (absolute velocity corresponds to four-velocity in usual terminology).

For an absolute velocity $\mathbf{u}$, we define the three dimensional spacelike linear subspace

$$\mathbf{E_u} := \{\mathbf{x} \in \mathbf{M} \mid \mathbf{u} \cdot \mathbf{x} = 0\}; \tag{1}$$

then

$$\boldsymbol{\pi}_\mathbf{u} := 1 + \mathbf{u} \otimes \mathbf{u} : \mathbf{M} \to \mathbf{E_u}, \quad \mathbf{x} \mapsto \mathbf{x} + \mathbf{u}(\mathbf{u} \cdot \mathbf{x}) \tag{2}$$

is the projection onto $\mathbf{E_u}$ along $\mathbf{u}$. The restriction of the Lorentz form onto $\mathbf{E_u}$ is positive definite, so $\mathbf{E_u}$ is a Euclidean vector space (this will correspond to the space vectors of an inertial observer with velocity $\mathbf{u}$).

The history of a classical material point is described by a differentiable *world line function* $r : \mathbf{I} \to M$ such that $\dot{r}(\mathbf{s})$ is an absolute velocity for all proper time values $\mathbf{s}$. The range of a world line function – a one dimensional submanifold – is called a *world line*.

An *observer* $\mathbf{U}$ is an absolute velocity valued smooth map defined in a connected open subset of M. (This is just a mathematical definition; it may sound unfamiliar at first, but considering that something that an observer calls a 'fixed space-point' is, in fact, a world line in spacetime, this definition will make perfect 'physical' sense). A maximal integral curve of $\mathbf{U}$ – a world line – is a *space point* of the observer, briefly a $\mathbf{U}$-*space point*; the set of the maximal integral curves of $\mathbf{U}$ is the *space* of the observer, briefly the $\mathbf{U}$-*space*.

An observer having constant value is called *inertial*. An inertial observer will be referred to by its constant velocity. The space points – the integral curves – of an inertial observer with absolute velocity $\mathbf{u}$ are straight lines parallel to $\mathbf{u}$. The $\mathbf{u}$-space point containing the world point x is the straight line $x + \mathbf{uI}$, where $\mathbf{uI} := \{\mathbf{ut} \mid \mathbf{t} \in \mathbf{I}\}$.

In order to arrive at the analogue of the coordinate system corresponding to an inertial observer we need to specify the time-syncronization of the observer. Of course, the standard syncroniztion is used: according to the standard synchronization of $\mathbf{u}$, two world points x and y are simultaneous if and only if $\mathbf{u} \cdot (y - x) = 0$. Thus, simultaneous world points form a hyperplane parallel to $\mathbf{E_u}$; such a hyperplane is an $\mathbf{u}$-*instant*, their set is $\mathbf{u}$-*time*. The $\mathbf{u}$-instant containing the world point x is the hyperplane $x + \mathbf{E_u}$.

An inertial observer together with its standard synchronization is called a *standard inertial frame*. Note that a standard inertial frame is an exactly defined object in our framework, it does not refer to any coordinates, coordinate axes, it contains an inertial observer and its standard synchronization only.

The space vector between two $\mathbf{u}$-space points (straight lines in spacetime) is the world vector between $\mathbf{u}$-simultaneous world points of the straight lines in question; in formula, $\mathbf{u}$-space, endowed with the subtraction

$$(x + \mathbf{uI}) - (y + \mathbf{uI}) := \boldsymbol{\pi}_\mathbf{u}(x - y) \tag{3}$$

becomes a three dimensional affine space over $\mathbf{E_u}$ (this fact shows that $\mathbf{E_u}$ does indeed correspond to the space vectors of the observer $\mathbf{u}$).

This is a crucial point: the space vectors of the standard inertial frame $\mathbf{u}$ are elements of $\mathbf{E_u}$, so *the space vectors of different inertial frames form different three dimensional vector spaces.*

The time passed between to $\mathbf{u}$-instants (hyperplanes in spacetime) is the time passed between them in an arbitrary $\mathbf{u}$-space point. In formula, $\mathbf{u}$-time, endowed with the subtraction

$$(x + \mathbf{E_u}) - (y + \mathbf{E_u}) := -\mathbf{u} \cdot (x - y) \tag{4}$$

becomes a one dimensional affine space over $\mathbf{I}$.

If r is a world line function, then the standard inertial frame with velocity value $\dot{r}(\mathbf{s})$ is called the *local rest frame corresponding to r at* $\mathbf{s}$.

In usual treatments the coordinates distinguish a certain inertial frame (the 'rest' frame) and any other inertial frame is considered through its relative velocity with respect to the rest frame (and the coordinates with respect to the new frame are given via the corresponding Lorentz transformation). The main feature of our approach is the systematic use of absolute velocities for characterizing standard inertial frames (this perfectly reflects the principle of relativity: no inertial frame can be distinguished compared to other inertial frames). Among several advantages, such as clarity of many concepts appearing in the theory of relativity, it often results in highly simplified and clear formulae.

3 Relative velocity and relative acceleration

Let r be a world line function (describing the history of a classical material point). A standard inertial frame with absolute velocity $\mathbf{u}$ gives a correspondence between $\mathbf{u}$-time t and the proper time $\mathbf{s}$ of the world line function r: if t_0 is the $\mathbf{u}$-instant of the world point $r(0)$, then, according to (4), $\mathbf{t} := (r(\mathbf{s}) + \mathbf{E_u}) - (r(0) + \mathbf{E_u}) = -\mathbf{u} \cdot (r(\mathbf{s}) - r(0))$; therefore

$$\frac{d\mathbf{t}}{d\mathbf{s}} = -\mathbf{u} \cdot \dot{r}(\mathbf{s}). \tag{5}$$

As a consequence, the proper time, too, can be given as a function of **u**-time, and

$$\frac{d\mathbf{s}}{d\mathbf{t}} = \frac{1}{-\mathbf{u}\cdot\dot{r}(\mathbf{s}(\mathbf{t}))}. \tag{6}$$

The inertial frame observes the history of the material point as a motion, assigning **u**-space points to **u**-instants : $r_{\mathbf{u}}(\mathbf{t}) := r(\mathbf{s}(\mathbf{t})) + \mathbf{u}\mathbf{I}$. Then, according to (3) and the previous equality, the relative velocity is (for the sake of brevity we omit the variable $\mathbf{t}$ from the expressions)

$$\mathbf{v_u} := r'_{\mathbf{u}} = \lim_{\mathbf{h}\to 0}\frac{r_{\mathbf{u}}(\mathbf{t}+\mathbf{h}) - r_{\mathbf{u}}(\mathbf{t})}{\mathbf{h}} = \frac{\dot{r}(\mathbf{s})}{-\mathbf{u}\cdot\dot{r}(\mathbf{s})} - \mathbf{u} \tag{7}$$

and the relative acceleration is

$$\mathbf{a_u} := r''_{\mathbf{u}} = \frac{1}{(-\mathbf{u}\cdot\dot{r}(\mathbf{s}))^2}\left(\ddot{r}(\mathbf{s}) + \frac{\dot{r}(\mathbf{s})(\mathbf{u}\cdot\ddot{r}(\mathbf{s}))}{-\mathbf{u}\cdot\dot{r}(\mathbf{s})}\right) \tag{8}$$

where the derivative according to **u**-time is denoted by a prime.

It is worth mentioning that

$$-\mathbf{u}\cdot\dot{r}(\mathbf{s}) = \frac{1}{\sqrt{1-|\mathbf{v_u}|^2}} =: \gamma_{\mathbf{u}}, \tag{9}$$

the well-known relativistic factor.

4 Lorentz boosts and discrete Thomas rotations

As we emphasized, the space vectors of different standard inertial frames form different three dimensional vector spaces; for the absolute velocities $\mathbf{u}$ and $\mathbf{u}'$, $\mathbf{E_u}$ and $\mathbf{E}'_{\mathbf{u}}$ are different vector spaces. A natural correspondence can be given between them, the *Lorentz boost* from $\mathbf{u}$ to $\mathbf{u}'$ (Matolcsi 1993, Matolcsi & Goher 2001),

$$\mathbf{B}(\mathbf{u}',\mathbf{u}) := 1 + \frac{(\mathbf{u}'+\mathbf{u})\otimes(\mathbf{u}'+\mathbf{u})}{1-\mathbf{u}'\cdot\mathbf{u}} - 2\mathbf{u}'\otimes\mathbf{u} \tag{10}$$

which is a Lorentz form preserving linear map on $\mathbf{M}$, such that $\mathbf{B}(\mathbf{u}',\mathbf{u})\mathbf{u} = \mathbf{u}'$. This is the absolute form (which appears implicitly in Rowe 1984, too) of the usual Lorentz boost. It is clear from the given formula that this absolute form depends on two absolute velocities. The explicit matrix form of a textbook Lorentz boost depends on a single relative velocity but, in fact, it also refers to two inertial observers (one of which is the 'rest frame', not appearing explicitly in the formulae).

The vector $\mathbf{q}'$ in the space of the inertial frame $\mathbf{u}'$ is called *physically equal* to the vector $\mathbf{q}$ in the space of the inertial frame $\mathbf{u}$ if $\mathbf{q}' = \mathbf{B}(\mathbf{u}', \mathbf{u})\mathbf{q}$; we say also that $\mathbf{q}$ *boosted* from $\mathbf{E_u}$ to $\mathbf{E}'_\mathbf{u}$ equals $\mathbf{q}'$. This Lorentz boost gives sense to the usual tacit assumption that the corresponding coordinate axes of different inertial frames are parallel. The coordinate axes defined by the vectors $\mathbf{e}_i$ in $\mathbf{E_u}$ are parallel to the axes defined by the vectors $\mathbf{e}'_i$ in $\mathbf{E}'_\mathbf{u}$ if $\mathbf{e}'_i = \mathbf{B}(\mathbf{u}', \mathbf{u})\mathbf{e}_i$ ($i = 1, 2, 3$). (The parallelism of frame axes is usually a nagging problem in standard treatments; see the discussion in the Introduction of Kennedy 2002.)

To be physically equal is a symmetric relation: $\mathbf{B}(\mathbf{u}', \mathbf{u})^{-1} = \mathbf{B}(\mathbf{u}, \mathbf{u}')$, so if $\mathbf{q}'$ is physically equal to $\mathbf{q}$, then $\mathbf{q}$ is physically equal to $\mathbf{q}'$.

On the other hand, to be physically equal is *not transitive*: the product of two Lorentz boosts, in general, is not a Lorentz boost (as it is well known): we have

$$\mathbf{B}(\mathbf{u}'', \mathbf{u}')\mathbf{B}(\mathbf{u}', \mathbf{u}) = \mathbf{B}(\mathbf{u}'', \mathbf{u}) \quad \text{iff} \quad \mathbf{u}, \mathbf{u}', \mathbf{u}'' \quad \text{are coplanar,} \tag{11}$$

(which is equivalent to the standard formalism: the relative velocity of $\mathbf{u}''$ with respect to $\mathbf{u}$ and the relative velocity of $\mathbf{u}'$ with respect to $\mathbf{u}$ are collinear.)

In an equivalent formulation,

$$\mathbf{R_u}(\mathbf{u}', \mathbf{u}'') := \mathbf{B}(\mathbf{u}, \mathbf{u}'')\mathbf{B}(\mathbf{u}'', \mathbf{u}')\mathbf{B}(\mathbf{u}', \mathbf{u}) \tag{12}$$

is the identity transformation if and only if $\mathbf{u}, \mathbf{u}', \mathbf{u}''$ are coplanar. Note that $\mathbf{R_u}(\mathbf{u}', \mathbf{u}'')\mathbf{u} = \mathbf{u}$ and the restriction of $\mathbf{R_u}(\mathbf{u}', \mathbf{u}'')$ onto $\mathbf{E_u}$ is a rotation, called the *discrete Thomas rotation corresponding to* $\mathbf{u}$*,* $\mathbf{u}'$ *and* $\mathbf{u}''$. The general formula for this rotation (in this absolute formalism) was given in Matolcsi &

Goher 2001.

Thus if $\mathbf{q}'$ is physically equal to $\mathbf{q}$ and $\mathbf{q}''$ is physically equal to $\mathbf{q}'$, then $\mathbf{q}$ need not be physicllay equal to $\mathbf{q}''$. This is why the Thomas rotation appears to be 'paradoxic'.

In other words, a vector $\mathbf{q}$ boosted from $\mathbf{E_u}$ to $\mathbf{E'_u}$ yields $\mathbf{q}'$ and then $\mathbf{q}'$ boosted from $\mathbf{E'_u}$ to $\mathbf{E_{u''}}$ yields $\mathbf{q}''$, and lastly $\mathbf{q}''$ boosted from $\mathbf{E_{u''}}$ back to $\mathbf{E_u}$, results in a vector rotated from the original $\mathbf{q}$.

5 Compasses

A boost, as defined above, does not mean a real transport of vectors from an observer space into another one. Nevertheless, it can be related to such a transport in the following situation.

A **compass** (a needle fixed to a central point) can be described in spacetime as a vector attached to a material point; more precisely, as a pair of functions $(r, \mathbf{z})$ where r is a world line function (the history of the material point) and $\mathbf{z}$ is a vector valued function (describing the direction of the needle) defined on the proper time of r, $\mathbf{z} : \mathbf{I} \to \mathbf{M}$, such that

– it is always spacelike according to the corresponding local rest frame of the world line, i.e. $\dot{r} \cdot \mathbf{z} = 0$,

– the magnitude of $\mathbf{z}$, $|\mathbf{z}|$ is constant.

Thus the needle of the compass passes continuously from the space of one local rest frame to that of another one. The compass is conceived to be **locally inertial** if $\mathbf{z}$ is physically constant along r (keeps direction in itself) i.e. the values of $\mathbf{z}$ are boosted continuously corresponding to the absolute velocities of the world line. This means that if $\mathbf{h}$ is a "small" time period, then $\mathbf{z}(\mathbf{s}+\mathbf{h})$ in $\mathbf{E}_{\dot{r}(\mathbf{s}+\mathbf{h})}$ is "nearly" physically equal to $\mathbf{z}(\mathbf{s})$ in $\mathbf{E}_{\dot{r}(\mathbf{s})}$, more precisely

$$\lim_{\mathbf{h}\to 0} \frac{\mathbf{z}(\mathbf{s}+\mathbf{h}) - \mathbf{B}(\dot{r}(\mathbf{s}+\mathbf{h}), \dot{r}(\mathbf{s}))\mathbf{z}(\mathbf{s})}{\mathbf{h}} = 0. \tag{13}$$

Because $\dot{r}\cdot\mathbf{z}=0$, we can replace $\big(\dot{r}(\mathbf{s}+\mathbf{h})+\dot{r}(\mathbf{s})\big)\cdot\mathbf{z}(\mathbf{s})$ with $\big(\dot{r}(\mathbf{s}+\mathbf{h})-\dot{r}(\mathbf{s})\big)\cdot\mathbf{z}(\mathbf{s})$, so

$$\mathbf{B}(\dot{r}(\mathbf{s}+\mathbf{h}),\dot{r}(\mathbf{s}))\mathbf{z}(\mathbf{s})=\mathbf{z}(\mathbf{s})+\frac{\big(\dot{r}(\mathbf{s}+\mathbf{h})+\dot{r}(\mathbf{s})\big)\big(\dot{r}(\mathbf{s}+\mathbf{h})-\dot{r}(\mathbf{s})\big)\cdot\mathbf{z}(\mathbf{s})}{1-\dot{r}(\mathbf{s}+\mathbf{h})\cdot\dot{r}(\mathbf{s})} \tag{14}$$

and the above limit becomes $\dot{\mathbf{z}}-\dot{r}(\ddot{r}\cdot\mathbf{z})=0$, from which, taking into account again $\dot{r}\cdot\mathbf{z}=0$, we get the well known Fermi-Walker equation along r

$$\dot{\mathbf{z}}=\dot{r}(\ddot{r}\cdot\mathbf{z})-\ddot{r}(\dot{r}\cdot\mathbf{z})=(\dot{r}\wedge\ddot{r})\mathbf{z}. \tag{15}$$

Note that the Lorentz boosts in terms of absolute velocities yielded this equation in an extremely brief and simple way (in contrast to the usual deductions, see e.g. Møller 1972).

If $\mathbf{z}$ is any vector satisfying the Fermi-Walker equation along r, then $(\dot{r}\cdot\mathbf{z})^{\cdot}=0$, so $\dot{r}\cdot\mathbf{z}$ is constant; if $\mathbf{z}(\mathbf{s}_0)$ is spacelike according to $\dot{r}(\mathbf{s}_0)$ for one proper time value $\mathbf{s}_0$, then $\mathbf{z}(\mathbf{s})$ is spacelike according to $\dot{r}(\mathbf{s})$ for all $\mathbf{s}$ ($\mathbf{z}$ is always spacelike according to the corresponding local rest frame of r). Moreover, then $\dot{\mathbf{z}}\cdot\mathbf{z}=0$, so the magnitude of $\mathbf{z}$ is constant.

Let us introduce another term. Let r be world line function. We call a function $\mathbf{z}:\mathbf{I}\to\mathbf{M}$ a **gyroscopic vector on** r if $\mathbf{z}$ satisfies the Fermi-Walker equation along r and a value of $\mathbf{z}$ is spacelike according to the corresponding local rest frame of r. Obviously, if $\mathbf{z}$ is a gyroscopic vector along r, then $(r,\mathbf{z})$ is a locally inertial compass. It is well known and easily verifiable that if $\mathbf{z}_1$ and $\mathbf{z}_2$ are gyroscopic vectors on the same world line, then $\mathbf{z}_1\cdot\mathbf{z}_2$ is constant (which corresponds to the fact that 'non-rotating' vectors retain their relative angle).

6 Circular world line

Take a standard inertial frame with velocity value $\mathbf{u}_c$. A circular motion with respect to this frame can be given by

– its centre q_c in $\mathbf{u}_c$-space,

– its angular velocity, an antisymmetric linear map $0 \neq \mathbf{\Omega} : \mathbf{E}_{\mathbf{u}_c} \to \frac{\mathbf{E}_{\mathbf{u}_c}}{\mathbf{I}}$ (usually one considers angular velocity as a spatial axial vector which, in fact, corresponds to an antisymmetric tensor),

– its initial position with respect to the centre, a vector $0 \neq \mathbf{q}$ in $\mathbf{E}_{\mathbf{u}_c}$, orthogonal to the kernel of $\mathbf{\Omega}$ such that $|\mathbf{\Omega}\mathbf{q}| < 1$.

This motion has the form

$$\mathbf{t} \mapsto q_c + e^{\mathbf{t}\mathbf{\Omega}}\mathbf{q} = q_c + \mathbf{q}\cos\omega\mathbf{t} + \frac{\mathbf{\Omega}\mathbf{q}}{\omega}\sin\omega\mathbf{t} \tag{16}$$

where $\omega := |\mathbf{\Omega}| = \sqrt{\frac{1}{2}\mathrm{Tr}\mathbf{\Omega}^*\mathbf{\Omega}}$. Note that we have

$$\mathbf{\Omega}^2\mathbf{q} = -\omega^2\mathbf{q}, \qquad |\mathbf{\Omega}\mathbf{q}| = \omega\rho \tag{17}$$

where $\rho := |\mathbf{q}|$.

The relative velocity of this motion equals $e^{\mathbf{t}\mathbf{\Omega}}\mathbf{\Omega}\mathbf{q}$ which has the magnitude $\omega\rho$. Thus, we infer from (5) and (9), that the relation between the proper time $\mathbf{s}$ of the world line and the $\mathbf{u}_c$-time $\mathbf{t}$ is $\mathbf{t} = \mathbf{s}\lambda$, where

$$\lambda := \frac{1}{\sqrt{1 - \omega^2\rho^2}}. \tag{18}$$

Then it is easy to see that this motion comes from the world line function

$$\mathbf{s} \mapsto r(\mathbf{s}) = o + \mathbf{s}\lambda\mathbf{u}_c + e^{\mathbf{s}\lambda\mathbf{\Omega}}\mathbf{q} \tag{19}$$

where o is a world point of the centre q_c (which is a straight line in spacetime). Then

$$\dot{r}(\mathbf{s}) = \lambda(\mathbf{u}_c + e^{\mathbf{s}\lambda\mathbf{\Omega}}\mathbf{\Omega}\mathbf{q}), \qquad \ddot{r}(\mathbf{s}) = -\lambda^2\omega^2 e^{\mathbf{s}\lambda\mathbf{\Omega}}\mathbf{q}. \tag{20}$$

Note that $\mathbf{u}_c$ is the absolute velocity of the centre and $\mathbf{u}_0 := \lambda(\mathbf{u}_c + \mathbf{\Omega}\mathbf{q})$ is the "initial" absolute velocity of the world line.

7 Gyroscopic vectors on a circular world line

Introducing the variable $\mathbf{t} := \lambda\mathbf{s}$ ($\mathbf{u}_c$-time) and the function $\hat{\mathbf{z}}(\mathbf{t}) := \mathbf{z}(\mathbf{t}/\lambda)$, then omitting the "hat" for brevity, we get the Fermi-Walker diferential equation (15) along the above circular world line in the form

$$\mathbf{z}'(\mathbf{t}) = -\lambda^2\omega^2\big((\mathbf{u}_c + e^{\mathbf{t}\mathbf{\Omega}}\mathbf{\Omega}\mathbf{q}) \wedge (e^{\mathbf{t}\mathbf{\Omega}}\mathbf{q})\big)\mathbf{z}(\mathbf{t}). \tag{21}$$

In the sequel we find it convenient to consider $\mathbf{\Omega}$ as defined on the whole of $\mathbf{M}$ in such a way that $\mathbf{\Omega}\mathbf{u}_c = 0$. Then $\mathbf{\Omega}$ will be a Lorentz antisymmetric linear map on the whole of $\mathbf{M}$, thus $e^{\mathbf{t}\mathbf{\Omega}}$ will preserve the Lorentz form (it will be a Lorentz transformation) for which $e^{\mathbf{t}\mathbf{\Omega}}\mathbf{u}_c = \mathbf{u}_c$ holds.

Then we infer that $\mathbf{a}(\mathbf{t}) := e^{-\mathbf{t}\mathbf{\Omega}}\mathbf{z}(\mathbf{t})$ satisfies

$$\mathbf{a}'(\mathbf{t}) = -\mathbf{\Omega}\mathbf{a}(\mathbf{t}) + e^{-\mathbf{t}\mathbf{\Omega}}\left(-\lambda^2\omega^2\big((\mathbf{u}_c + e^{\mathbf{t}\mathbf{\Omega}}\mathbf{\Omega}\mathbf{q}) \wedge (e^{\mathbf{t}\mathbf{\Omega}}\mathbf{q})\big)\right)e^{\mathbf{t}\mathbf{\Omega}}\mathbf{a}(\mathbf{t}). \tag{22}$$

Using the fact that $e^{-\mathbf{t}\mathbf{\Omega}}\mathbf{u}_c = \mathbf{u}_c$ and the tensorial identity that the transposed of the term $e^{\mathbf{t}\mathbf{\Omega}}$ on the right of the tensor product can be carried into the second factor of the product, we arrive at the autonomouos linear differential equation

$$\mathbf{a}'(\mathbf{t}) = -\mathbf{\Gamma}\mathbf{a}(\mathbf{t}) \tag{23}$$

where

$$\mathbf{\Gamma} := \mathbf{\Omega} + \lambda^2\omega^2(\mathbf{u}_c + \mathbf{\Omega}\mathbf{q}) \wedge \mathbf{q} = \lambda^2\mathbf{\Omega} + \lambda^2\omega^2\mathbf{u}_c \wedge \mathbf{q} \tag{24}$$

where the latter equality relies on the simple fact that

$$(\mathbf{\Omega}\mathbf{q}) \wedge \mathbf{q} = \rho^2\mathbf{\Omega}. \tag{25}$$

As a consequence – since $\mathbf{a}(0) = \mathbf{z}(0)$ –, we get the solution of the Fermi-Walker differential equation in the form

$$\mathbf{z}(\mathbf{t}) = e^{\mathbf{t}\mathbf{\Omega}}e^{-\mathbf{t}\mathbf{\Gamma}}\mathbf{z}(0). \tag{26}$$

Let us investigate the properties of

$$\mathbf{F}(\mathbf{t}) := e^{\mathbf{t}\mathbf{\Omega}}e^{-\mathbf{t}\mathbf{\Gamma}} \tag{27}$$

which we call the *Fermi-Walker operator* at $\mathbf{t} = \lambda\mathbf{s}$, $\mathbf{s}$ being a proper time point of the circular world line function.

Since $\mathbf{\Gamma}$ is an antisymmetric linear map, $e^{-\mathbf{t}\mathbf{\Gamma}}$ is a Lorentz transformation. It is trivial that $\mathbf{\Gamma}\mathbf{u}_0 = 0$, thus the restriction of $e^{-\mathbf{t}\mathbf{\Gamma}}$ onto the three dimensional Euclidean space $\mathbf{E}_{\mathbf{u}_0}$ is a rotation.

We know that the restriction of the Lorentz transformation $e^{\mathbf{t}\mathbf{\Omega}}$ onto the Euclidean vector space $\mathbf{E}_{\mathbf{u}_c}$ is a rotation, and its restriction onto $E_{\mathbf{u}_0}$ is an orthogonal transformation onto $\mathbf{E}_{\dot{r}(\mathbf{t}/\lambda)}$.

Thus $e^{\mathbf{t}\mathbf{\Omega}}e^{-\mathbf{t}\mathbf{\Gamma}}$, as a product of two Lorentz transformations, is a Lorentz transformation, too. Its restriction onto $\mathbf{E}_{\mathbf{u}_0}$ is a Euclidean structure preserving linear bijection from $\mathbf{E}_{\mathbf{u}_0}$ onto $\mathbf{E}_{\dot{r}(\mathbf{t}/\lambda)}$. This can be conceived as a spatial rotation *only if* $\dot{r}(\mathbf{t}/\lambda) = \mathbf{u}_0$ (otherwise it acts between different Euclidean spaces).

8 Thomas rotation on the circular world line

The absolute velocity of the circular world line is periodic, $\dot{r}(\frac{2\pi}{\omega\lambda}) = \dot{r}(0) = \mathbf{u}_0$. Since $e^{\frac{2\pi}{\omega}\mathbf{\Omega}}$ is the identity map, we have for the corrresponding Fermi-Walker operator

$$\mathbf{F}\left(\frac{2\pi}{\omega}\right) = e^{-\frac{2\pi}{\omega}\mathbf{\Gamma}} \tag{28}$$

whose restriction onto the Euclidean vector space $\mathbf{E}_{\mathbf{u}_0}$ is a rotation, called the *Thomas rotation on the circular world line* (19).

The angle of the Thomas rotation is $2\pi - \frac{2\pi}{\omega}|\mathbf{\Gamma}|$ where $|\mathbf{\Gamma}|$ is the magnitude of $\mathbf{\Gamma}$; $\mathrm{Tr}\mathbf{\Gamma}^*\mathbf{\Gamma} = \lambda^4\mathrm{Tr}(-\mathbf{\Omega}^2 - \omega^4\mathbf{q}\otimes\mathbf{q} + \omega^4\rho^2\mathbf{u}_c\otimes\mathbf{u}_c) = \lambda^4(2\omega^2 - 2\omega^4\rho^2) = 2\lambda^2\omega^2$, therefore $|\mathbf{\Gamma}| = \lambda\omega$.

As a consequence, the Thomas angle on the circular world line equals

$$2\pi\left(1 - \frac{1}{\sqrt{1-\omega^2\rho^2}}\right) \tag{29}$$

which is the well known result (Thomas 1927).

It is worth noting that the value of a gyroscopic vector after a whole revolution equals the original one if and only if the gyroscopic vector is parallel to the kernel of $\mathbf{\Omega}$ i.e. is orthogonal to the plane of rotation in the space of the centre.

9 Generalizations

Besides deriving the Thomas angle on the circular world line in a short and transparent way, our method gives the Thomas rotation operator itself, and allows us the following insight into the behaviour of gyroscopic vectors in general.

Let r be an arbitrary world line function. The solutions of the corresponding Fermi-Walker equation with various initial values give us a Fermi-Walker operator $\mathbf{F}(\mathbf{s}_2, \mathbf{s}_1)$, a Lorentz transformation for all proper time points $\mathbf{s}_1$ and $\mathbf{s}_2$ such that

$$\dot{r}(\mathbf{s}_2) = \mathbf{F}(\mathbf{s}_2, \mathbf{s}_1)\dot{r}(\mathbf{s}_1) \tag{30}$$

and

$$\mathbf{z}(\mathbf{s}_2) = \mathbf{F}(\mathbf{s}_2, \mathbf{s}_1)\mathbf{z}(\mathbf{s}_1) \tag{31}$$

for an arbitrary gyroscopic vector $\mathbf{z}$ on r.

Thus the restriction of $\mathbf{F}(\mathbf{s}_2, \mathbf{s}_1)$ onto $\mathbf{E}_{\dot{r}(\mathbf{s}_1)}$ – the space vectors of the local rest frame at $\mathbf{s}_1$ – is a Euclidean structure preserving linear bijection onto $\mathbf{E}_{\dot{r}(\mathbf{s}_2)}$ – the space vectors of the local rest frame at $\mathbf{s}_2$.

In particular, if $\dot{r}(\mathbf{s}_2) = \dot{r}(\mathbf{s}_1)$, the restriction of $\mathbf{F}(\mathbf{s}_2, \mathbf{s}_1)$ onto $\mathbf{E}_{\dot{r}(\mathbf{s}_1)}$ is a rotation, which we call the *Thomas rotation on the world line r, corresponding to the proper time points $\mathbf{s}_1$ and $\mathbf{s}_2$*.

It is worth noting: *a Thomas rotation on a world line for two proper time values has a meaning only if the corresponding absolute velocites are equal.* Thus no Thomas rotation can be defined on a world line if all its absolute velocity values are different.

10 Thomas precession with respect to an inertial frame

Now, let $\mathbf{z}$ be a gyroscopic vector on the world line function r. An inertial frame $\mathbf{u}$ observes $\mathbf{z}$ by boosting it continuously to its own space, i.e. giving the function $\mathbf{z_u} : \mathbf{I} \to \mathbf{E_u}$ such that

$$\mathbf{z_u}(\mathbf{t}) := \mathbf{B}(\mathbf{u}, \dot{r}(\mathbf{s}(\mathbf{t})))\mathbf{z}(\mathbf{s}(\mathbf{t})). \tag{32}$$

Then, omitting $\mathbf{t}$ as previously, we infer that

$$\begin{aligned}
\mathbf{z}'_{\mathbf{u}} &= \frac{1}{-\mathbf{u}\cdot\dot{r}(\mathbf{s})}\Big(\Big(\frac{d}{d\mathbf{s}}\mathbf{B}(\mathbf{u},\dot{r}(\mathbf{s}))\Big)\mathbf{z}(\mathbf{s}) + \mathbf{B}(\mathbf{u},\dot{r}(\mathbf{s}))\dot{\mathbf{z}}(\mathbf{s})\Big) \\
&= \frac{1}{-\mathbf{u}\cdot\dot{r}(\mathbf{s})}\Big(\Big(\frac{d}{d\mathbf{s}}\mathbf{B}(\mathbf{u},\dot{r}(\mathbf{s}))\Big)\mathbf{B}(\dot{r}(\mathbf{s}),\mathbf{u})\mathbf{z_u} + \\
&\qquad \mathbf{B}(\mathbf{u},\dot{r}(\mathbf{s}))(\dot{r}(\mathbf{s})\wedge\ddot{r}(\mathbf{s}))\mathbf{B}(\dot{r}(\mathbf{s}),\mathbf{u})\mathbf{z_u}\Big)
\end{aligned} \tag{33}$$

Consider the second term: using the fact that $\mathbf{B}(\mathbf{u}, \dot{r}(\mathbf{s}))\dot{r}(\mathbf{s}) = \mathbf{u}$ and the tensorial identity that the transposed of the term $\mathbf{B}(\dot{r}(\mathbf{s}), \mathbf{u})$ can be carried into the second factor of the product, we get (omitting $\mathbf{s}$ for the sake of brevity)

$$\mathbf{B}(\mathbf{u},\dot{r})(\dot{r}\wedge\ddot{r})\mathbf{B}(\dot{r},\mathbf{u})\mathbf{z_u} = \mathbf{u}\wedge\mathbf{B}(\mathbf{u},\dot{r})\ddot{r} = \mathbf{u}\wedge\left(\ddot{r} + \frac{\dot{r}(\mathbf{u}\cdot\ddot{r})}{1-\mathbf{u}\cdot\dot{r}}\right) \tag{34}$$

where in the second equality we applied the expression (10) . As concerns the first term, a straightforward calculation (using formula (10)) yields that it equals

$$\frac{\dot{r}\wedge\ddot{r}}{1-\mathbf{u}\cdot\dot{r}} - \mathbf{u}\wedge\ddot{r} - 2\mathbf{u}\wedge\left(\frac{\dot{r}(\mathbf{u}\cdot\ddot{r})}{1-\mathbf{u}\cdot\dot{r}}\right) \tag{35}$$

Taking into account (7) and (8), we finally obtain

$$\mathbf{z}'_{\mathbf{u}} = \frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{v_u}\wedge\mathbf{a_u})\mathbf{z_u} \tag{36}$$

which is the known result.

Thus the inertial frame $\mathbf{u}$ sees the gyroscopic vector $\mathbf{z}$ – which keeps direction in itself – precessing, the angular velocity of precession is the antisymmetric linear map (depending on $\mathbf{u}$-time)

$$\Omega_{\mathbf{u}} := \frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}\mathbf{v}_{\mathbf{u}} \wedge \mathbf{a}_{\mathbf{u}} = \frac{\gamma_{\mathbf{u}}-1}{|\mathbf{v}_{\mathbf{u}}|^2}\mathbf{v}_{\mathbf{u}} \wedge \mathbf{a}_{\mathbf{u}} : \mathbf{E}_{\mathbf{u}} \to \frac{\mathbf{E}_{\mathbf{u}}}{\mathbf{I}}. \tag{37}$$

We call the reader's attention to the fact: *the same gyroscopic vector precesses differently with respect to different inertial frames.*

11 Thomas precessions corresponding to a circular world line

Let us consider the circular world line described in Section 6.

Let us take the standard inertial frame of the centre i.e. the one with absolute velocity $\mathbf{u}_c$. Then equalities in (20), (7) and (8) yield

$$\mathbf{v}_{\mathbf{u}_c}(\mathbf{t}) = e^{\mathbf{t}\boldsymbol{\Omega}}\boldsymbol{\Omega}\mathbf{q}, \qquad \mathbf{a}_{\mathbf{u}_c}(\mathbf{t}) = -\omega^2 e^{\mathbf{t}\boldsymbol{\Omega}}\mathbf{q}. \tag{38}$$

Then $\mathbf{v}_{\mathbf{u}_c}(\mathbf{t}) \wedge \mathbf{a}_{\mathbf{u}_c}(\mathbf{t}) = -\omega^2 e^{\mathbf{t}\boldsymbol{\Omega}}((\boldsymbol{\Omega}\mathbf{q}) \wedge \mathbf{q})e^{-\mathbf{t}\boldsymbol{\Omega}} = -\omega^2\rho^2\boldsymbol{\Omega}$ because of (25). Since $\omega^2\rho^2 = |\mathbf{v}_{\mathbf{u}_c}|^2$, the angular velocity of the Thomas precession with respect to the "central frame" $\mathbf{u}_c$ is constant in $\mathbf{u}_c$-time, equalling

$$\left(1 - \frac{1}{\sqrt{1-\omega^2\rho^2}}\right)\boldsymbol{\Omega}. \tag{39}$$

Usual treatments (Møller 1972) consider exclusively this precession in connection with the circular world line, i.e. the Thomas precession with respect to the central frame. Of course, there are other possibilities, too.

For instance, let us take the standard inertial frame in which the gyroscopic vector is at rest initially i.e. the one with absolute velocity $\mathbf{u}_0 = \lambda(\mathbf{u}_c + \boldsymbol{\Omega}\mathbf{q})$.

Then

$$\gamma_{\mathbf{u}_0}(\mathbf{s}) = -\mathbf{u}_0 \cdot \dot{r}(\mathbf{s}) = \lambda^2(1 - \omega^2\rho^2\cos\omega\lambda\mathbf{s}). \tag{40}$$

Consequently, solving (5) we get that the $\mathbf{u}_0$-time $\mathbf{t}$ and the proper time $\mathbf{s}$ have the relation $\mathbf{t} = \lambda^2\mathbf{s} - \lambda\omega^2\rho^2\sin\omega\lambda\mathbf{s}$. Then in view of (7), we find

$$\mathbf{v}_{\mathbf{u}_0} = \lambda\frac{(\mathbf{u}_c + e^{\lambda\mathbf{s}\boldsymbol{\Omega}}\boldsymbol{\Omega}\mathbf{q})(1-\omega^2\rho^2)}{1-\omega^2\rho^2\cos\omega\lambda\mathbf{s}} - \lambda(\mathbf{u}_c + \boldsymbol{\Omega}\mathbf{q}) \tag{41}$$

and a similar, more complicated formula gives $\mathbf{a}_{\mathbf{u}_0}$, too; as a consequence, the angular velocity of the Thomas precession with respect to the inertial frame $\mathbf{u}_0$ depends rather intricately on $\mathbf{u}_0$-time. For instance, if n is an arbitrary natural number, then

– for $\mathbf{u}_0$-instants given by $\lambda\mathbf{s} = \frac{2n\pi}{\omega}$, the value of the relative velocity is zero, so the angular velocity of Thomas precession has zero value, too;

– for $\mathbf{u}_0$-instants given by $\lambda\mathbf{s} = \frac{(2n-1)\pi}{\omega}$, the relative velocity equals $-\frac{2\lambda}{1+\omega^2\rho^2}(\omega^2\rho^2\mathbf{u}_c + \mathbf{\Omega q})$ and the relative acceleration is $\frac{(1-\omega^2\rho^2)\omega^2}{(1+\omega^2\rho^2)^2}\mathbf{q}$, so the angular velocity of Thomas precession has value

$$-\frac{\lambda}{(1+\omega^2\rho^2)\rho^2}(\omega^2\rho^2\mathbf{u}_c + \mathbf{\Omega q}) \wedge \mathbf{q} = -\frac{1}{\lambda(1+\omega^2\rho^2)}\mathbf{\Gamma} \tag{42}$$

with $\mathbf{\Gamma}$ as in (24). The magnitude of this instantenuous angular velocity is $\frac{\omega}{1+\omega^2\rho^2}$.

12 Discussion

The systematic use of absolute velocities instead of relative ones gives us a nice form of the Lorentz boosts which results in extremely brief and simple derivation of

– the discrete Thomas rotation due to successive Lorentz boosts (see Matolcsi & Goher 2001 for the explicit formula),

– the Fermi-Walker equation,

– the Thomas rotation on a circular world line,

– the concept of Thomas rotations in general,

– the Thomas precession with respect to an inertial frame,

– a clear conceptual distinction between Thomas rotation and Thomas precession.

It is an important fact that the Thomas rotation is absolute i.e. independent of reference frames, while the Thomas precession is relative i.e. refers to inertial frames. It is emphasized again that the same gyroscope shows different precessions when observed by different inertial frames.

In a forthcoming paper we use a similar treatment in the same absolute formalism to show that the change of polarization of light when reflected is an analogous phenomenon to the discrete Thomas rotation.

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