SUMSETS AND THE CONVEX HULL

MÁTÉ MATOLCSI AND IMRE Z. RUZSA

Abstract. We extend Freiman's inequality on the cardinality of the sumset of a d dimensional set. We consider different sets related by an inclusion of their convex hull, and one of them added possibly several times.

1. Introduction

The aim of this paper is to give a lower estimate for the cardinality of certain sumsets in $\mathbb{R}^d$.

We say that a set in $\mathbb{R}^d$ is *proper d-dimensional* if it is not contained in any affine hyperplane.

Our starting point is the following classical theorem of Freiman.

Theorem 1.1 (Freiman[1], Lemma 1.14)**.** *Let $A \subset \mathbb{R}^d$ be a finite set, $|A| = m$. Assume that A is proper d-dimensional. Then*

$$|A + A| \geq m(d+1) - \frac{d(d+1)}{2}.$$

We will show that to get this inequality it is sufficient to use the vertices (extremal points) of A.

Definition 1.2. We say that a point $a \in A$ is a *vertex* of a set $A \subset \mathbb{R}^d$ if it is not in the convex hull of $A \setminus \{a\}$. The set of vertices will be denoted by $\operatorname{vert} A$.

The convex hull of a set A will be denoted by $\operatorname{conv} A$.

Theorem 1.3. *Let $A \subset \mathbb{R}^d$ be a finite set, $|A| = m$. Assume that A is proper d-dimensional, and let $A' = \operatorname{vert} A$, We have*

$$|A + A'| \geq m(d+1) - \frac{d(d+1)}{2}.$$

This can be extended to different summands as follows.

Theorem 1.4. *Let $A, B \subset \mathbb{R}^d$ be finite sets, $|A| = m$. Assume that B is proper d-dimensional and $A \subset \operatorname{conv} B$. We have*

$$|A + B| \geq m(d+1) - \frac{d(d+1)}{2}.$$

Finally we extend it to several summands as follows. We use $kB = B + \cdots + B$ to denote repeated addition. As far as we know even the case of $A = B$ seems to be new here.

2000 *Mathematics Subject Classification.* 11B50, 11B75, 11P70.

Supported by Hungarian National Foundation for Scientific Research (OTKA), Grants No. PF-64061, T-049301, T-047276.

Supported by Hungarian National Foundation for Scientific Research (OTKA), Grants No. K 61908, K 72731.

Theorem 1.5. *Let $A, B \subset \mathbb{R}^d$ be finite sets, $|A| = m$. Assume that B is proper d-dimensional and $A \subset \operatorname{conv} B$. Let k be a positive integer. We have*

$$|A + kB| \geq m\binom{d+k}{k} - k\binom{d+k}{k+1} = \left(m - \frac{kd}{k+1}\right)\binom{d+k}{k}. \tag{1.1}$$

The case $d = 1$ of the above theorems is quite obvious. In [2] we gave a less obvious result which compares a complete sum and its subsums, which sounds as follows.

Theorem 1.6. *Let $A_1, \dots, A_k$ be finite, nonempty sets of integers. Let A_i' be the set consisting of the smallest and the largest elements of A_i (so that $1 \leq |A_i'| \leq 2$). Put*

$$S = A_1 + \dots + A_k,$$

$$S_i = A_1 + \dots + A_{i-1} + A_{i+1} + \dots + A_k,$$

$$S_i' = A_1 + \dots + A_{i-1} + A_i' + A_{i+1} + \dots + A_k,$$

$$S' = \bigcup_{i=1}^{k} S_i'.$$

We have

$$|S| \geq |S'| \geq \frac{1}{k-1}\sum_{i=1}^{k}|S_i| - \frac{1}{k-1}. \tag{1.2}$$

Problem 1.7. Generalize Theorem 1.6 to multidimensional sets. A proper generalization should give the correct order of magnitude, hence the analog of (1.2) could be of the form

$$|S| \geq |S'| \geq \left(\frac{k^{d-1}}{(k-1)^d} - \varepsilon\right)\sum_{i=1}^{k}|S_i|$$

if all sets are sufficiently large.

Problem 1.8. Let $A, B_1, \dots, B_k \subset \mathbb{R}^d$ such that the B_i are proper d-dimesional and

$$A \subset \operatorname{conv} B_1 \subset \operatorname{conv} B_2 \subset \dots \subset \operatorname{conv} B_k.$$

Does the esimate given in (1.1) also hold for $A + B_1 + \dots + B_k$?

This is easy for $d = 1$.

2. A simplicial decomposition

We will need a result about simplicial decomposition.

By a *simplex* in $\mathbb{R}^d$ we mean a proper d-dimensional compact set which is the convex hull of $d + 1$ points.

Definition 2.1. Let $S_1, S_2 \subset \mathbb{R}^d$ be simplices, $B_i = \operatorname{vert} S_i$. We say that they are in *regular position*, if

$$S_1 \cap S_2 = \operatorname{conv}(B_1 \cap B_2),$$

that is, they meet in a common k-dimensional face for some $k \leq d$. (This does not exclude the extremal cases when they are disjoint or they coincide.) We say that a collection of simplices is in regular position if any two of them are.

Lemma 2.2. *Let $B \subset \mathbb{R}^d$ be a proper d dimensional finite set, $S = \operatorname{conv} B$. There is a sequence $S_1, S_2, \dots, S_n$ of distinct simplices in regular position with the following properties.*

a) $S = \bigcup S_i$.

b) $B_i = \operatorname{vert} S_i = S_i \cap B$.

c) Each S_i, $2 \le i \le n$ meets at least one of $S_1, \dots, S_{i-1}$ in a $(d-1)$ dimensional face.

We mentioned this lemma to several geometers and all answered "of course" and offered a proof immediately, but none could name a reference with this formulation, so we include a proof for completeness. This proof was communicated to us by prof. Károly Böröczki.

Proof. We use induction on $|B|$. The case $|B| = 2$ is clear. Let $|B| = k$, and assume we know it for smaller sets (in any possible dimension).

Let b be a vertex of B and apply it for the set $B' = B \setminus \{b\}$. This set may be d or $d-1$ dimensional.

First case: B' is d dimensional. With the natural notation let

$$S' = \bigcup_{i=1}^{n'} S'_i$$

be the prescribed decomposition of $S' = \operatorname{conv} B'$. We start the decomposition of S with these, and add some more as follows.

We say that a point x of S' is *visible* from b, if x is the only point of the segment joining x and b in S'. Some of the simplices S'_i have (one or more) $d-1$ dimensional faces that are completely visible from b. Now if F is such a face, then we add the simplex

$$\operatorname{conv}(F \cup \{b\})$$

to our list.

Second case: B' is $d-1$ dimensional. Again we start with the decomposition of S', just in this case the sets S'_i will be $d-1$ dimensional simplices. Now the decomposition of S will simply consist of

$$S_i = \operatorname{conv}(S'_i \cup \{b\}),\ n = n'.$$

□

The construction above immediately gave property c). We note that it is not really an extra requirement, every decomposition has it after a suitable rearrangement. This just means that the graph obtained by using our simplices as vertices and connecting two of them if they share a $d-1$ dimensional face is connected. Now take two simplices, say S_i and S_j. Take an inner point in each and connect them by a segment. For a generic choice of these point this segment will not meet any of the $\le d-2$ dimensional faces of any S_k. Now as we walk along this segment and go from one simplex into another, this gives a path in our graph between the vertices corresponding to S_i and S_j.

3. The case of a simplex

Here we prove Theorem 1.5 for the case $|B| = d + 1$.

Lemma 3.1. *Let $A, B \subset \mathbb{R}^d$ be finite sets, $|A| = m$, $|B| = d+1$. Assume that B is proper d-dimensional and $A \subset \operatorname{conv} B$. Let k be a positive integer. Write $|A \cap B| = m_1$. We have*

$$|A + kB| = (m - m_1)\binom{d+k}{k} + \binom{d+k+1}{k+1} - \binom{d-m_1+k+1}{k+1}. \tag{3.1}$$

In particular, if $|A \cap B| \le 1$, then

$$|A + kB| = m\binom{d+k}{k}. \tag{3.2}$$

We have always

$$|A + kB| \ge m\binom{d+k}{k} - k\binom{d+k}{k+1} = \left(m - \frac{kd}{k+1}\right)\binom{d+k}{k}. \tag{3.3}$$

Proof. Put $A_1 = A \cap B$, $A_2 = A \setminus B$. Write $B = \{b_0, \dots, b_d\}$, arranged in such a way that

$$A_1 = A \cap B = \{b_0, \dots, b_{m_1-1}\}.$$

The elements of kB are the points of the form

$$s = \sum_{i=0}^{d} x_i b_i, \ \ x_i \in \mathbb{Z}\ , x_i \ge 0, \ \sum x_i = k,$$

and this representation is unique. Clearly

$$|kB| = \binom{d+k}{k}.$$

Each element of A has a unique representation of the form

$$a = \sum_{i=0}^{k} \alpha_i d_i, \ \ \alpha_i \in \mathbb{R}\ , \alpha_i \ge 0, \ \sum \alpha_i = 1,$$

$$a = \sum_{i=0}^{d} \alpha_i b_i, \ \ \alpha_i \in \mathbb{R}\ , \alpha_i \ge 0, \ \sum \alpha_i = 1,$$

and if $a \in A_1$, then some $\alpha_i = 1$ and the others are equal to 0, while if $a \in A_2$, then at least two α_i's are positive.

Assume now that $a + s = a' + s'$ with certain $a, a' \in A$, $s, s' \in kB$. By substituting the above representations we obtain

$$\sum (\alpha_i + x_i) b_i = \sum (\alpha_i' + x_i') b_i, \ \sum (\alpha_i + x_i) = \sum (\alpha_i' + x_i') = k+1,$$

hence $\alpha_i + x_i = \alpha_i' + x_i'$ for all i. By looking at the integral and fractional parts we see that this is possible only if $\alpha_i = \alpha_i'$, or one of them is 1 and the other is 0. If the second possibility never happens, then $a = a'$. If it happens, say $\alpha_i = 1, \alpha_i' = 0$ for some i, then $\alpha_j = 0$ for all $j \ne i$ and then each a_j' must also be 0 or 1, that is, $a, a' \in A_1$.

The previous discussion shows that $(A_1 + kB) \cap (A_2 + kB) = \emptyset$ and the sets $a + kB$, $a \in A_2$ are disjoint, hence

$$|A + kB| = |A_1 + kB| + |A_2 + kB|$$

and

$$|A_2 + kB| = |A_2|\,|kB| = (m - m_1)\binom{d+k}{k}. \tag{3.4}$$

Now we calculate $|A_1 + kB|$. The elements of this set are of the form

$$\sum_{i=0}^{d} x_i b_i, \; x_i \in \mathbb{Z}\ , x_i \geq 0, \; \sum x_i = k+1,$$

with the additional requirement that there is at least one subscript i, $i \leq m_1 - 1$ with $x_i \geq 1$. Without this requirement the number would be the same as

$$|(k+1)B| = \binom{d+k+1}{k+1}.$$

The vectors $(x_0, \dots, x_d)$ that violate this requirement are those that use only the last $d - m_1$ coordinates, hence their number is

$$\binom{d - m_1 + k + 1}{k+1}.$$

We obtain that

$$|A_1 + kB| = \binom{d+k+1}{k+1} - \binom{d - m_1 + k + 1}{k+1}.$$

Adding this formula to (3.4) we get (3.1).

If $m_1 = 0$ or 1, this formula reduces to the one given in (3.2).

To show inequality (3.3), observe that this formula is a decreasing function of m_1, hence the minimal value is at $m_1 = d+1$, which after an elementary transformation corresponds to the right side of (3.3). Naturally this is attained only if $m \geq d+1$, and for small values of m the right side of (3.3) may even be negative. □

4. The general case

Proof of Theorem 1.5. We apply Lemma 2.2 to our set B. This decomposition induces a decomposition of A as follows. We put

$$A_1 = A \cap S_1, A_2 = A \cap (S_2 \setminus S_1), \dots, A_n = A \cap \big(S_n \setminus (S_1 \cup S_2 \cup \dots \cup S_{n-1})\big).$$

Clearly the sets A_i are disjoint and their union is A. Recall the notation $B_i = \operatorname{vert} S_i$.

We claim that the sets $A_i + kB_i$ are also disjoint.

Indeed, suppose that $a + s = a' + s'$ with $a \in A_i$, $a' \in A_j$, $s \in kB_i$, $s' \in kB_j$, $i < j$. We have

$$\frac{a+s}{k+1} \in S_i, \; \frac{a'+s'}{k+1} \in S_j,$$

and these points are equal, so they are in

$$S_i \cap S_j = \operatorname{conv}(B_i \cap B_j).$$

This means that in the unique convex representation of $(a' + s')/(k+1)$ by points of B_j only elements of $B_i \cap B_j$ are used. However, we can obtain this representation via using the representation of a' and the components of s', hence we must have $a' \in \operatorname{conv}(B_i \cap B_k) \subset S_i$, a contradiction.

This disjointness yields

$$|A+kB| \geq \sum |A_i+kB_i| \,.$$

We estimate the summands using Lemma 3.1.

If $i>1$, then $|A_i\cap B_i|\leq 1$. Indeed, there is a $j<i$ such that S_j has a common $d-1$ dimensional face with S_i, and then the d vertices of this face are excluded from A_i by definition. So in this case (3.2) gives

$$|A_i+kB_i| = |A_i| \binom{d+k}{k}.$$

For $i=1$ we can only use the weaker estimate (3.3):

$$|A_1+kB_1| \geq |A_1| \binom{d+k}{k} - k\binom{d+k}{k+1}.$$

Summing these equations we obtain (1.1). □

Acknowledgement. The authors profited much from discussions with Katalin Gyarmati and Károly Böröczky. In particular, we are indebted to Prof. Böröczky for the present version of the proof of Lemma 2.2.

References

[1] G. Freiman, *Foundations of a structural theory of set addition*, American Math. Soc., 1973.

[2] Katalin Gyarmati, M. Matolcsi, and I. Z. Ruzsa, *A superadditivity and submultiplicativity property for cardinalities of sumsets*, Combinatorica, to appear.

Alfréd Rényi Institute of Mathematics, Budapest, Pf. 127, H-1364 Hungary, (also at BME Department of Analysis, Budapest, H-1111, Egry J. u. 1)

E-mail address: matomate@renyi.hu

Alfréd Rényi Institute of Mathematics, Budapest, Pf. 127, H-1364 Hungary

E-mail address: ruzsa@renyi.hu