Practise exercises 1.

(1) Prove that if $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$, then A is Jordan measurable and $\lambda_J(A) = 0$.

(2) Prove that for any unbounded set $A \subset \mathbb{R}^d$ we have $\lambda_J^*(A) = +\infty$.

(3) Let $A = \mathbb{Q} \cap [0,1]$. Prove that $\lambda_J^*(A) = 1$ but $\lambda_{*,J}(A) = 0$, so A is not Jordan measurable.

(4) Prove that the fat Cantor set C_F is not Jordan measurable.

(5) Prove that we don't get anything new if we allow countable union in the definition of the inner measure. That is, $\sup\{\sum_{n=1}^{\infty} \lambda(T_n) \ : \ \dot\cup T_n \subset A\} = \lambda_{*,J}(A)$.

(6) (HW1) Prove that if $\alpha : S \to [0, +\infty]$ is finitely additive on a semi-ring S then α is monotone, σ-superadditive, and subadditive.

(7) Prove that a σ-algebra is closed under countable intersections.

(8) Let $\alpha : S \to [0, +\infty]$ be finitely additive on a semi-ring S, and let α^* denote the corresponding outer measure. Prove that $\alpha^*(A) = \inf\{\sum_{i=1}^{\infty} \alpha(A_i) \ : \ A_i \in S, A \subset \dot\cup A_i\}$.

(9) Prove that all open sets are Lebesgue measurable.

(10) Prove that the Borel σ-algebra is strictly contained in the σ-algebra of Lebesgue-measurable sets. That is, $(\mathbb{R}^d, \mathcal{B}, \lambda) \subsetneq (\mathbb{R}^d, \mathcal{M}(\lambda^*), \lambda)$.

Practise exercises 2.

(11) (HW1) Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E, F \in \mathcal{M}$. Prove that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

(12) Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E \in \mathcal{M}$. Define $\mu_E(A) = \mu(A \cap E)$ for any $A \in \mathcal{M}$. Prove that μ_E is a measure.

(13) Let $A_1, \dots, A_r$ be sets belonging to a semi-ring S. Prove that there exist some pairwise disjoint sets $B_1, \dots, B_m$ in S such that $\cup_{i=1}^r A_i = \dot\cup_{j=1}^m B_j$.

(14) (HW1) Prove the Borel-Cantelli lemma: "if A_i are events such that the sum of their probability is finite, then the probability that infinitely many of them occurs is 0". More formally, if μ is a probability measure on a σ-algebra $\mathcal{A}$, and $A_i \in \mathcal{A}$ are such that $\sum_{i=1}^{\infty} \mu(A_i) < +\infty$ then $\mu(\cap_{n \in \mathbb{N}} \cup_{k \geq n} A_k) = 0$. (Note here that the event $\cap_{n \in \mathbb{N}} \cup_{k \geq n} A_k$ describes exactly that infinitely many of the A_i's occur.)

(15) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$, and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{A}, F \subset N \text{ for some } N \in \mathcal{N}\}$. Prove that $\overline{\mathcal{M}}$ is a σ-algebra.

(16) In the setting of the previous exercise, let $\overline{\mu}(E \cup F) = \mu(E)$. Prove that $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$. (The measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ is called the *completion* of $(X, \mathcal{M}, \mu)$.

(17) (HW1) Let μ be a finite Borel measure on $\mathbb{R}$. Let $F(x) = \mu((-\infty, x])$. Prove that F is an increasing and right-continuous function. (F is called the *distribution function* of μ.)

(18) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function. Let $\mu((a, b]) = F(b) - F(a)$ for every $a \leq b$. Prove that μ is finitely additive and σ-subadditive on the semi-ring of intervals $S = \{(a, b] : a \leq b\}$. (The corresponding measure via the Caratheodory extension is called the *Lebesgue-Stieltjes measure* associated to F.)

(19) Prove that the Lebesgue-measure is *open-regular*, i.e. $\lambda(E) = \inf\{\lambda(U) : U \supset E, U \text{is open}\}$.

(20) Prove that the Lebesgue-measure is *compact-regular*, i.e. $\lambda(E) = \sup\{\lambda(K) : K \subset E, K \text{is compact}\}$.

Practise exercises 3.

(21) Assume that the σ-algebra $\mathcal{A}$ is generated by some set system $\mathcal{E}$. Prove that $f : (X, \mathcal{M}) \to (Y, \mathcal{A})$ is $(\mathcal{M}, \mathcal{A})$-measurable iff $f^{-1}(E) \in \mathcal{M}$ for every $E \in \mathcal{E}$.

(22) Prove that $f : (X, \mathcal{A}) \to \mathbb{C}$ is measurable (i.e. Borel-measurable), iff $Re\ f$ and $Im\ f$ are measurable.

(23) Prove that if $f, g : X \to \mathbb{C}$ are measurable, then $f + g$ and fg are also measurable.

(24) Prove that if (f_n) is a sequence of real-valued measurable functions on $(X, \mathcal{A})$ then the functions $g_1(x) = \sup_{n \in \mathbb{N}} f_n(x)$, $g_2(x) = \inf_{n \in \mathbb{N}} f_n(x)$, $g_3(x) = \limsup_{n \in \mathbb{N}} f_n(x)$ and $g_4(x) = \liminf_{n \in \mathbb{N}} f_n(x)$ are all measurable. (In particular, if $\lim_{n \in \mathbb{N}} f_n(x)$ exists for every $x \in X$ then the limit function is also measurable.)

(25) Prove the monotone convergence theorem: if f_n is an increasing sequence of $[0, +\infty]$-valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$ then $\int_X (\lim_{n \in \mathbb{N}} f_n) d\mu = \lim_{n \in \mathbb{N}} \int_X f_n d\mu$.

(26) (HW2) Prove that for $f \in L^+$ we have $\int_X f d\mu = 0$ iff $f(x) = 0$ for μ-almost every $x \in X$.

(27) (HW2) Let $f = (f_1, f_2, \ldots, f_n)$ be a function from $(X, \mathcal{A})$ to $\mathbb{R}^n$. Prove that f is Borel-measurable iff all f_j are Borel-measurable.

(28) Prove that if $f, g : (X, \mathcal{A}) \to \mathbb{R}^n$ are Borel-measurable functions then $h(x) = \langle f(x), g(x) \rangle$ is also Borel-measurable.

(29) Let $(X, \mathcal{A})$, $(Y, \mathcal{M})$ be measurable spaces and let $A \in \mathcal{A} \otimes \mathcal{M}$ be a measurable set in the product space. Show that the cross-sections $A_x = \{y \in Y : (x, y) \in A\}$ and $A_y = \{x \in X : (x, y) \in A\}$ are measurable for each $x \in X$ and $y \in Y$.

(30) (HW2) Let $(X, \mathcal{A})$ be a measurable space, and $H_1, \dots, H_n \subset X$ pairwise disjoint sets. Assume that there exist some numbers $a_1, \dots, a_n \in \mathbb{R}$ such that the function $f = \sum_{j=1}^{n} a_j \chi_{H_j}$ is measurable. Does this imply that each H_j must be a measurable set?

(31) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Show that f is measurable.

(32) (HW2) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that f is measurable.

Practise exercises 4.

(33) (HW3) Let $f_1, f_2 \in L^+$. Using the definition of the integral, prove that $\int f_1 + f_2 = \int f_1 + \int f_2$.

(34) Let $f \in L^1$. Prove that $|\int f| \leq \int |f|$.

(35) (HW3) Assume (f_n) is a sequence of functions in L^1 such that $\sum_{n=1}^{\infty} \|f_n\|_1 < +\infty$. Prove that $\sum_{n=1}^{\infty} f_n$ converges almost everywhere, and $\int (\sum_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} \int f_n$.

(36) (HW3) Let $f_n : (X, \mathcal{A}, \mu) \to \mathbb{R}_+$ be a monotonically decreasing sequence of functions, and assume that $\int f_1 < +\infty$. Prove that $\int (\lim_{n\to\infty} f_n) = \lim_{n\to\infty} \int f_n$. Give an example where $\int f_1 = +\infty$ and the above equality is not true.

(37) Calculate $\lim_{n\to\infty} \int_0^n e^{-2x}(1 + \frac{x}{n})^n dx$.

(38) (HW3) Calculate $\lim_{n\to\infty} \int_1^n \frac{\pi + 2x^2 \arctan(nx)}{x^4(2-2^{-n}) + \sin\frac{1}{n}}$.

(39) Prove that the function $F(t) = \int_0^\infty e^{-tx} dx$ is infinitely many times differentiable on $(0, +\infty)$, and calculate $\int_0^\infty x^n e^{-x} dx$.

(40) Prove that if $f_n \to f$ in L^1 then $f_n \to f$ in measure. Show by an example that the converse is not true.

(41) Assume that $f_n \to f$ in measure. Prove that there exists a subsequence f_{n_j} such that $f_{n_j} \to f$ almost everywhere.

(42) Prove that if $f_n \to f$ in L^1 then there exists a subsequence f_{n_j} such that $f_{n_j} \to f$ almost everywhere.

(43) Prove Egoroff's theorem: if $\mu(X) < +\infty$ and $f_n \to f$ almost everywhere, then for every $\varepsilon > 0$ there exists a set $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

Practise exercises 5.

(44) Let μ the counting measure on $\mathbb{N}$. Prove that $f_n \to f$ in measure iff $f_n \to f$ uniformly.

(45) Prove that $f_n \to f$ in measure iff for every $\varepsilon > 0$ there exists N such that $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$ for all $n \geq N$.

(46) Assume $f_n \to f$ and $g_n \to g$ in measure. Prove that $f_n + g_n \to f + g$ in measure. Prove that $f_n g_n \to fg$ in measure if $\mu(X) < +\infty$ but not necessarily if $\mu(X) = +\infty$.

(47) Prove that if μ is σ-finite and $f_n \to f$ a.e., then there exist measurable sets $E_1, E_2, \ldots, \subset X$ such that$\mu((\cup_{j=1}^{\infty} E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j. (Hint: Use Egoroff's theorem.)

(48) Assume $\mu(X) <= \infty$, For complex-valued measurable functions let $d(f,g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu$. Prove that this defines a metric if we identify functions that are equal almost everywhere. Prove also, that $f_n \to f$ in this metric iff $f_n \to f$ in measure.

(49) Prove Lusin's theorem: If $f : [a,b] \to \mathbb{C}$ is measurable and $\varepsilon > 0$, then there exists a compact set $E \subset [a,b]$ such that $\lambda(E^c) < \varepsilon$ and $F|_E$ is continuous. (Here λ denotes the Lebesgue measure.)

(50) Compute the following integral: $\int_{x=0}^{1} \int_{y=x}^{1} x \frac{\sinh y}{y} dy dx$.

(51) Prove that $\int_0^1 x^a (1-x)^{-1} \log x = \sum_{k=1}^{\infty} (a+k)^{-2}$ for all $a > -1$.

(52) For $x \in \mathbb{R}^n$ let $f(x) = g(|x|)$ for some nonnegaive measurable function g defined on $(0,\infty)$. Prove that $\int_{\mathbb{R}^n} f(x) dx = \sigma(S^{n-1}) \int_0^{\infty} \int g(r) r^{n-1} dr$. (Here $\sigma(S^{n-1})$ denotes the surface area of the unit sphere.)

(53) Prove that $\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$.

(54) Prove that $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

(55) For which values of a, b is the function $|x|^a |\log|x||^b$ integrable on the ball of radius 2 in $\mathbb{R}^n$?

(56) In a σ-finite measure space $(X, \mathcal{A}, \mu)$, for $f \in L^+(X)$ let $G_f = \{(x,y) \in X \times [0,\infty] : y \leq f(x)\}$. Prove that G_f is $\mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$-measurable, and $(\mu \times \lambda)(G_f) = \int_X f d\mu$. (Here λ is the Lebesgue measure.)

(57) (HW4) Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, $f \in L^1(\mu), g \in L^1(\nu)$, and let $h(x,y) = f(x)g(y)$. Prove that $\int h d(\mu \times \nu) = (\int f d\mu)(\int g d\nu)$.

(58) This exercise shows that σ-finiteness of the measures is needed in the Fubini theorem. Let κ and λ denote the counting measure and the Lebesgure measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$, respectively. Let $f(x,y) = 1$ if $0 \le x = y \le 1$, and $f(x,y) = 0$ otherwise. Compute $\int_{\mathbb{R}\times\mathbb{R}} f d(\kappa \times \lambda)$, $\int_{\mathbb{R}}(\int_{\mathbb{R}} f d\kappa)d\lambda$, $\int_{\mathbb{R}}(\int_{\mathbb{R}} f d\lambda)d\kappa$.

Practise exercises 6.

(59) (HW4) Let $f \in L^1$, $g \in L^\infty$. Prove that $\|fg\|_1 \le \|f\|_1\|g\|_\infty$.

(60) Prove that $\|f\|_\infty = \inf\{a \ge 0 : \mu(\{x : |f(x)| > a\}) = 0\}$ defines a norm on L^∞.

(61) Prove that $\|f_n - f\|_\infty \to 0$ iff there exists a measurable set E such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.

(62) (HW4) Show by examples that $L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R})$ for any $1 < p \ne q < \infty$.

(63) Prove that for any $1 < p \le q < \infty$ we have $\ell^p \subset \ell^q$, and for any sequence $\underline{a} = (a_n)_{n\in\mathbb{N}} \in \ell^p$ we have $\|\underline{a}\|_q \le \|\underline{a}\|_p$.

(64) (HW4) Assume $\mu(X) < +\infty$. Prove that for any $1 < p < q < \infty$ we have $L^p(X) \supset L^q(X)$, and $\|f\|_p \le \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

(65) In the setting of the previous exercise, prove that $\lim_{p\to\infty} \|f\|_p \to \|f\|_\infty$.

(66) Prove that $L^p(\mathbb{R})$ is separable for $1 \le p < \infty$, but not separable for $p = \infty$.

(67) For a complex-valued measurable function f define the essential range R_f of f as the set of complex numbers z such that $\{x : |f(x) - z| < \varepsilon\}$ has positive measure for every $\varepsilon > 0$. Prove that R_f is a closed set, and if $f \in L^\infty$ then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

(68) Let p, q be conjugate exponents (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), and let $h \in L^q(X)$. Define a linear functional $\phi_h : L^p(X) \to \mathbb{C}$ by setting $\phi_h(f) = \int_X f(x)h(x)d\mu(x)$. Prove that ϕ_h is a bounded linear functional, and $\|\phi_h\| \le \|h\|_q$.

(69) Assume $1 \le p < \infty$ and $\|f_n - f\|_p \to 0$. Prove that $f_n \to f$ in measure. On the other hand, prove also that if $f_n \to f$ in measure, and $|f_n| \le g \in L^p$ then $\|f_n - f\|_p \to 0$.

(70) Assume $1 \le p < \infty$, $f_n, f \in L^p$ and $f_n \to f$ almost everywhere. Prove that $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$.

Practise exercises 7.

(71) (HW5) Let ν be a signed measure on $(X, \mathcal{A})$. Prove that $E \in \mathcal{A}$ is ν-null iff $|\nu|(E) = 0$.

(72) Let ν be a signed measure and μ be a measure on $(X, \mathcal{A})$. Prove that $\nu \perp \mu$ iff $|\nu| \perp \mu$.

(73) Let ν be a signed measure on $(X, \mathcal{A})$. Prove that for all $f \in L^1(\nu)$ we have $|\int f d\nu| \leq \int |f| d|\nu|$.

(74) (HW5) Let ν be a signed measure on $(X, \mathcal{A})$. Prove that $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{A}, F \subset E\}$.

(75) Let ν be a signed measure on $(X, \mathcal{A})$. Prove that $|\nu|(E) = \sup\{\sum_{i=1}^n |\nu(E_i)| : E_i \in \mathcal{A}, \dot{\cup} E_i = E\}$.

(76) Prove that if ν is a signed measure, λ, μ are positive measures such that $\nu = \lambda - \mu$ then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

(77) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, g \in L^1(\mu)$. Prove that $f = g$ a.e. iff $\int_E f = \int_E g$ for all $E \in \mathcal{A}$.

(78) (HW5) Let ν be a signed measure and μ be a measure on $(X, \mathcal{A})$. Prove that $\nu \ll \mu$ iff $|\nu| \ll \mu$.

(79) Let μ and ν_j $(j = 1, 2, \dots)$ be positive measures such that $\nu_j \perp \mu$ for all j. Prove that $\sum_{j=1}^\infty \nu_j \perp \mu$.

(80) Let μ and ν_j $(j = 1, 2, \dots)$ be positive measures such that $\nu_j \ll \mu$ for all j. Prove that $\sum_{j=1}^\infty \nu_j \ll \mu$.

(81) (HW5) Let $F(x) = \arctan x$ if $x < 0$ and $F(x) = \frac{2x+1}{x+1}$ if $x \geq 0$. Let μ be the Lebesgue-Stieltjes measure generated by F, i.e. $\mu((a, b]) = F(b) - F(a)$. Find the Lebesgue decomposition (i.e the singular and absolutely continuous parts) of μ with respect to the Lebesgue measure of the real line.