Practise exercises 1.

(1) Prove that if $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ is a finite set, then A is Jordan measurable and $\lambda_J(A) = 0$.

(2) Prove that we don't get anything new if we allow countable union in the definition of the Jordan inner measure. That is, $\sup\{\sum_{n=1}^{\infty} \lambda(T_n) \ : \ \dot{\cup} T_n \subset A\} = \lambda_{*,J}(A)$.

(3) Let $A = \mathbb{Q} \cap [0,1]$. Prove that $\lambda_J^*(A) = 1$ but $\lambda_{*,J}(A) = 0$, so A is not Jordan measurable. Conclude that the Jordan measure is not σ-additive. (Therefore, it is *not* a measure. The terminology $Jordan - measure$ is standard, but let's keep in mind that it is not a measure.)

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E, F \in \mathcal{M}$. Show that $\mu(E)+\mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

(5) Let $(X, \mathcal{M}, \mu)$ be a measure space, amd $E \in \mathcal{M}$. Let $\mu_E(A) = \mu(A \cap E)$ for all $A \in \mathcal{M}$. Prove that μ_E is also a measure on the space $(X, \mathcal{M})$.

(6) Let X be a non-empty set, and let $A \subset X$. Determine the σ-algebra generated by
(a) $\{A\}$
(b) $\{B : B \subset A\}$

(7) Let $f : X \to Y$ be an arbitrary function. Prove that
(a) if $\mathcal{B}$ is a σ-algebra on Y then $\{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra on X.
(b) if $\mathcal{A}$ is a σ-algebra on X then $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y.

(8) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$, and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{A}, F \subset N \text{ forsome } N \in \mathcal{N}\}$. Prove that $\overline{\mathcal{M}}$ is also a σ-algebra.

(9) Let $(X, \mathcal{A})$, $(Y, \mathcal{M})$ be measurable spaces, and let $T \in \mathcal{A} \otimes \mathcal{M}$ be a measurable set in the product σ-algebra $\mathcal{A} \otimes \mathcal{M}$. Prove that the cross sections $T_x = \{y \in Y : (x, y) \in T\}$ and $T_y = \{x \in X : (x, y) \in T\}$ are measurable for all $x \in X$ and $y \in Y$.

Practise exercises 2.

(10) Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove the continuity properties of the measure:
a; If $E_1 \subset E_2 \subset \dots$, then $\mu(\cup_{j=1}^{\infty} = \lim_{j \to \infty} \mu(E_j)$
b; If $\mu(E_1) < \infty$, and $E_1 \supset E_2 \supset \dots$, then $\mu(\cap_{j=1}^{\infty} = \lim_{j \to \infty} \mu(E_j)$

(11) Prove the Borel-Cantelli lemma: "if A_i are events such that the sum of their probability is finite, then the probability that infinitely many of them occurs is 0". More formally, if μ is a probability measure on a σ-algebra $\mathcal{A}$, and $A_i \in \mathcal{A}$ are such that $\sum_{i=1}^{\infty} \mu(A_i) < +\infty$ then $\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k) = 0$. (Note here that the event $\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$ describes exactly that infinitely many of the A_i's occur.)

(12) Prove directly that the Lebesgue outer measure of $[0,1]$ is 1, and that it satisfies the splitting property. Conclude that $[0,1]$ is Lebesgue-measurable and satisfies $\lambda([0,1]) = 1$.

(13) Let $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x \geq 0$. Follow the construction of the Lebesgue-Stieltjes measure corresponding to F, and prove that it is equal to the Dirac measure δ_0.

(14) Let $F(x) = x$ if $x < 0$ and $F(x) = 1 + x$ if $x \geq 0$. Describe the Lebesgue-Stieltjes measure generated by F.

(15) Prove that the Lebesgue-measure is *open-regular*, i.e. $\lambda(E) = \inf\{\lambda(U) : U \supset E, U \text{is open}\}$.

(16) Let $A \subset \mathbb{R}^n$ be a set such that its Lebesgue outer measure is 0. Prove that A is Lebesgue measurable and $\lambda(A) = 0$.

(17) An example that the push-forward does not work in the naive way even for surjective functions: Let $X = \mathbb{R}$ and let $\mathcal{L}$ denote the collection of Lebesgue measurable sets, and let $E \notin \mathcal{L}$ be a non-measurable set. Let $c \notin E$ be any real number. Let $f(x) = x$ if $x \in E$ and $f(x) = c$ otherwise, and let $Y = E \cup \{c\}$. Then $f : X \to Y$ is surjective. Consider $\mathcal{A} = \{f(A) : A \in \mathcal{L}\}$. Then $\mathcal{A}$ is not a σ-algebra because $\{c\} \in \mathcal{A}$ but its complement in Y is E and it is not an image of a measurable set.

Practise exercises 3.

(18) Let $X = [0,1] \times [0,1]$ be the unit square, and let $\mathcal{E}$ be the collection of rectangles $T_{ab} = \{(x,y) \in X : 0 \leq a \leq x \leq b \leq 1, 0 \leq y \leq 1\}$. Let ν be the set function defined by $\nu : \mathcal{E} \to [0,1]$, $\nu(T_{ab}) = b - a$. Let μ^* be the outer measure generated by ν. Prove that the set $D = \{(x,y) \in X : x = y\}$ is not measurable (i.e. it does not satisfy the splitting property).

(19) Let $X, \mathcal{A})$, $(Y, \mathcal{M})$ be a measurable spaces. Assume $\mathcal{M}$ is generated by a set system $\mathcal{E}$. Prove that $f : X \to Y$ is $(\mathcal{A}, \mathcal{M})$ measurable iff $f^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{E}$.

(20) Let $X, \mathcal{A})$, $(Y_1, \mathcal{M}_1)$, $(Y_2, \mathcal{M}_2)$ be measurable spaces. Prove that a function $f = (f_1, f_2) : X \to Y_1 \times Y_2$ is $\mathcal{M}_1 \otimes \mathcal{M}_2$-measurable iff both coordinate functions f_1 and f_2 are measurable.

(21) Let $X, \mathcal{A})$ be a measurable space. Prove that a function $f : X \to \mathbb{C}$ is measurable iff Ref and Imf are measurable.

(22) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Prove that f is Borel-measurable.

(23) Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that for $f \in L^+$ we have $\int_X f d\mu = 0$ iff $f(x) = 0$ for μ-almost every $x \in X$.

(24) Let $f_1, f_2 \in L^+$. Using the definition of the integral, prove that $\int f_1 + f_2 = \int f_1 + \int f_2$.

(25) (HW1) Let $c+E = \{c+e : e \in E\}$ be a translated copy of a set $E \subset \mathbb{R}$. Prove that the Lebesgue outer measure λ^* is translation invariant, i.e. $\lambda^*(E) = \lambda^*(c+E)$ for every $E \subset \mathbb{R}$, $c \in \mathbb{R}$. Also, prove that if E is measurable (i.e. it has the splitting property) then $c+E$ is also measurable. Conclude that the Lebesgeu-measure is translation invariant.

(26) (HW2) In the setting of Exercise 8 define $\overline{\mu}(E \cup F) = \mu(E)$. Prove that $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$. (The space $(X, \overline{\mathcal{M}}, \overline{\mu})$ is called the *completion* of $(X, \mathcal{M}, \mu)$).

(27) (HW3) Prove that the Lebesgue-measure is *compact-regular*, i.e. $\lambda(E) = \sup\{\lambda(K) : K \subset E, K \text{is compact}\}$. (Hint: use Exercise 15.

(28) (HW4) Let $X, \mathcal{A})$ be a measurable space, and $f, g : X \to \mathbb{R}$ Borel-measurable functions. Prove that $f + g : X \to \mathbb{R}$ is also Borel-measurable.

Practise exercises 4.

(29) Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let $f, g : X \to \mathbb{R}$ be functions such that $f = g$ μ-almost everywhere. Prove that if f is measurable then so is g.

(30) Let $X = \mathbb{N}$, and μ the counting measure (i.e. $\mu(A) = |A|$ if A is finite, and $\mu(A) = \infty$ if A is infinite). Show that in this case L^+ is the set of nonnegative sequences, and for $f = (a_n) \in L^+$ we have $\int_X f d\mu = \sum_{n=1}^{\infty} a_n$.

(31) Let $(X, \mathcal{A})$ be a measurable space, and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \ldots$ be measures on it. Use the previous exercise and the monotone convergence theorem to prove that $\mu(E) = \sup_{n \in \mathbb{N}} \mu_n(E)$ is also a measure on $(X, \mathcal{A})$-n.

(32) Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \in L^+$, and $\lambda(E) = \int_E f d\mu$ for all $E \in \mathcal{A}$. Prove that λ is a measure on $(X, \mathcal{A})$. (In class we saw that this is so if f is a simple function. You can freely use this fact here.)

(33) Show an example where $f_n \in L^+$, $f_n(x)$ is monotonically decreasing, $f_n(x) \to f(x)$ for all $x \in X$, but $\int f \neq \lim_{n\to\infty} \int f_n$. Prove that if we assume $\int f_1 < \infty$ then $\int f = \lim_{n\to\infty} \int f_n$.

(34) Let $f \in L^1$ Prove that $|\int f| \leq \int |f|$. (Treat the cases of real-valued f and complex-valued f separately.)

(35) Let $f, g \in L^1$. Prove that $f + g \in L^1$.

(36) (HW1) Show that the Lebesgue measure of the Cantor set is 0.

(37) (HW2) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function such that $f(x) \neq 0$ for all x. Prove that the reciprocal function $1/f$ is also Borel measurable.

(38) (HW3) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that the derivative f' is Borel measurable.

(39) (HW4) Assume $f_n \in L^+$, $f \in L^+$, and $f_n(x)$ monotonically increases and converges to $f(x)$ for almost every x. Prove that $\int f = \lim_{n\to\infty} \int f_n$.

Practise exercises 5.

(40) (HW1) Expandig the term $\frac{1}{1-x}$ as an infinite series, and using the MCT prove that $\int_0^1 x^a(1-x)^{-1}\log x = -\sum_{k=1}^{\infty}(a+k)^{-2}$ for all $a > -1$.

(41) Assume (f_n) is a sequence of functions in L^1 such that $\sum_{n=1}^{\infty} \|f_n\|_1 < +\infty$. Prove that $\sum_{n=1}^{\infty} f_n$ converges almost everywhere, and $\int(\sum_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} \int f_n$.

(42) Calculate $\lim_{n\to\infty} \int_0^n e^{-2x}(1+\frac{x}{n})^n dx$.

(43) Calculate $\lim_{n\to\infty} \int_1^n \frac{\pi+2x^2\arctan(nx)}{x^4(2-2^{-n})+\sin\frac{1}{n}}$.

(44) (HW2) Calculate $\lim_{n\to\infty} \int_1^{\infty} \frac{n}{1+n^2x^2}dx$, and $\lim_{n\to\infty} \int_0^{\infty} \frac{n}{1+n^2x^2}dx$

(45) (HW3) Let $f_n(x) = e^{-n(1-\cos x)}$. Prove that on the interval $[0, 100]$ $f_n \to 0$ in measure (with respect to the Lebesgue measure), but on the half-line $[0, \infty)$ f_n does not converge to 0 in measure.

(46) Prove that $f_n \to f$ in measure iff for every $\varepsilon > 0$ there exists N such that $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$ for all $n \geq N$.

(47) Assume $f_n \to f$ and $g_n \to g$ in measure. Prove that $f_n + g_n \to f + g$ in measure.

(48) Assume $\mu(X) <= \infty$, For complex-valued measurable functions let $d(f, g) = \int_X \frac{|f-g|}{1+|f-g|}d\mu$. Prove that this defines a metric if we identify functions that are equal almost everywhere. Prove also, that $f_n \to f$ in this metric iff $f_n \to f$ in measure.

(49) Prove that if $f_n \to f$ almost uniformly then $f_n \to f$ in measure, and $f_n \to f$ almost everywhere.

(50) Let $\mu(X) < \infty$. Prove that if $f_n \to f$ uniformly, then $f_n \to f$ in L^1. Show an example where $\mu(X) < \infty$, $f_n \to f$ almost uniformly but $f_n \not\to f$ in L^1.

(51) Prove Lusin's theorem: If $f : [a, b] \to \mathbb{C}$ is measurable and $\varepsilon > 0$, then there exists a compact set $E \subset [a, b]$ such that $\lambda(E^c) < \varepsilon$ and $F|_E$ is continuous. (Here λ denotes the Lebesgue measure.)

(52) (HW4) Using Fubini's theorem compute the following integral: $\int_{x=0}^{1}\int_{y=x}^{1} x\frac{\sinh y}{y}dydx$.

(53) Let $(X,\mathcal{A},\mu),(Y,\mathcal{B},\nu)$ be σ-finite measure spaces, $f\in L^1(\mu), g\in L^1(\nu)$, and let $h(x,y)=f(x)g(y)$. Prove that $\int hd(\mu\times\nu)=(\int fd\mu)(\int gd\nu)$.

(54) This exercise shows that σ-finiteness of the measures is needed in the Fubini theorem. Let κ and λ denote the counting measure and the Lebesgure measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$, respectively. Let $f(x,y)=1$ if $0\le x=y\le 1$, and $f(x,y)=0$ otherwise. Compute $\int_{\mathbb{R}\times\mathbb{R}} fd(\kappa\times\lambda)$, $\int_{\mathbb{R}}(\int_{\mathbb{R}} fd\kappa)d\lambda$, $\int_{\mathbb{R}}(\int_{\mathbb{R}} fd\lambda)d\kappa$.

(55) Prove that continuous functions are dense in $L^1(\mathbb{R})$. That is, for any $f\in L^1(\mathbb{R})$ and any $\varepsilon>0$ there exists a continuous function g such that $\|f-g\|_1<\varepsilon$.

Practise exercises 6. (summary before the Midterm)

(56) Let $X_1,X_2,\dots$ be pairwise disjoint sets and for every n let $\mathcal{A}_n$ be a σ-algebra on X_n. Let $\mathcal{U}=\{\cup_{n=1}^{\infty}A_n : A_n\in\mathcal{A}_n\}$. Is it true that $\mathcal{U}$ is a σ-algebra on the set $X=\cup_{n=1}^{\infty}X_n$?

(57) Let $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x-y$. Determine the σ-algebra $\mathcal{A}=\{f^{-1}(E): E\in\mathcal{B}_{\mathbb{R}}\}$.

(58) Let $(X,\mathcal{A},\mu)$ be a measure space, and let $T:X\to Y$ be a surjective mapping. We saw that $\mathcal{B}=\{E\subset Y: T^{-1}(E)\in\mathcal{A}\}$ is a σ-algebra on Y. Prove that $\nu(E)=\mu(T^{-1}(E))$ is a measure on $(Y,\mathcal{B})$.

(59) Let μ^* be an outer measure on a set X, and let $B\subset A\subset X$ such that $\mu^*(A)=\mu^*(B)<\infty$. Prove that if B is measurable then so is A.

(60) Let $A\subset\mathbb{R}^n$ be a bounded set such that $\lambda(\mathrm{int}A)=\lambda(\overline{A})$ (where λ is the Lebesgue measure on $\mathbb{R}^n$, $\mathrm{int}A$ is the interior of A, and $\overline{A}$ is the closure of A. Prove that A is Lebesgue-measurable.

(61) Prove that if $f,g:\mathbb{R}\to\mathbb{R}^n$ are Borel-measurable functions, then $h(x)=\langle f(x),g(x)\rangle$ is also Borel-measurable (where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $\mathbb{R}^n$).

(62) Consider the measure space $(\mathbb{R},P(\mathbb{R}),\mu)$ where μ is the counting measure (i.e. $\mu(A)$ is the number of elements of A if A is finite, and $\mu(A)=+\infty$ if A is infinite). Describe the functions $f:\mathbb{R}\to\mathbb{R}$ which belong to $L^1(\mu)$.

(63) Prove that if $f_n\to f$ pointwise and there exists a $g\in L^1$ such that $|f_n|\le g$ for all n, then $f_n\to f$ in L^1.

(64) Calculate $\lim_{n\to\infty}\int_0^1\frac{nx^n}{1+x}dx$. (Hint: use integration by parts before LDCT).

(65) Let $f\in L^1(\mathbb{R})$, and let $E_1\subset E_2\subset\dots$ be Lebesgue-measurable sets in $\mathbb{R}$. Prove that $\lim_{n\to\infty}\int_{E_n} f$ exists and is finite.

(66) Is it true that the sequence of functions $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ converges in measure to $f(x) = e^x$ on the half-line $(-\infty, 0]$?

(67) Let $f, g \in L^1(\mathbb{R})$. Is it true that their product $fg \in L^1(\mathbb{R})$?

Practise exercises 7.

(68) Let f be Lebesgue integrable on $(0, a)$, and let $g(x) = \int_x^a \frac{f(t)}{t} dt$. Prove that g is also integrable on $(0, a)$ and $\int_0^a g(x)dx = \int_0^a f(x)dx$.

(69) Prove that $\int_0^\infty \frac{e^{-x}}{x} \sin x dx = \arctan 1$. (Hint: use Fubini's theorem for the function $e^{-xy} \sin x$.)

(70) For which values of a, b is the function $|x|^a| \log |x||^b$ integrable on the ball of radius $1/2$ in $\mathbb{R}^n$? (Use integration in polar coordinates.)

(71) Let $1 < p < \infty$. For a measurable function $f : X \to \mathbb{C}$ we say that $f \in L^p$ if $|f|^p \in L^1$, that is, $\int_X |f|^p < +\infty$. Show by examples that $L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R})$ for any $1 < p \neq q < \infty$.

(72) Assume $\mu(X) < +\infty$. Prove that for any $1 < p < q < \infty$ we have $L^p(X) \supset L^q(X)$.

(73) Assume $1 \leq p < \infty$ and $\|f_n - f\|_p \to 0$. Prove that $f_n \to f$ in measure. On the other hand, prove also that if $f_n \to f$ in measure, and $|f_n| \leq g \in L^p$ then $\|f_n - f\|_p \to 0$.

Practise exercises 8.

(74) (HW1) Let $f(x) = \frac{\cos x}{x^2+1}$, and for any Borel set E let $\nu(E) = \int_E f(x)dx$. What is $\|f\|_\infty$? Prove that ν is a signed measure, and determine the positive part P and the negative part N of the real line with respect to ν.

(75) Let $f(x) = \frac{\cos x}{|x|+1}$, and for any Borel set E let $\nu(E) = \int_E f(x)dx$. Is ν a signed measure? (Why not?)

(76) (HW2) Let ν be a signed measure on $(X, \mathcal{A})$. Prove that $E \in \mathcal{A}$ is ν-null iff $|\nu|(E) = 0$.

(77) Let ν be a signed measure and μ be a measure on $(X, \mathcal{A})$. We say that ν is *singular* with respect to μ (in notation: $\mu \perp \nu$) if there exist $E, F \in \mathcal{A}$ such that $E \cap F = \emptyset$, $E \cup = X$, and E is μ-null, and F is ν-null. Prove that $\nu \perp \mu$ iff $|\nu| \perp \mu$.

(78) Recall from class the Jordan decomposition $\nu = \nu^+ - \nu^-$. Prove that $\nu + \perp \nu^-$.

(79) Let μ and ν_j $(j = 1, 2, \dots)$ be positive measures such that $\nu_j \perp \mu$ for all j. Prove that $\sum_{j=1}^\infty \nu_j \perp \mu$.

(80) (HW3) Let ν be a signed measure. Recall that a set E is positive with respect to ν if for any measurable $F \subset E$ we have $\nu(F) \geq 0$. Prove that the countable union of positive sets is positive.

(81) Let ν be a signed measure on $(X, \mathcal{A})$. Prove that $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{A}, F \subset E\}$.

(82) Let ν be a signed measure on $(X, \mathcal{A})$. Prove that $|\nu|(E) = \sup\{\sum_{i=1}^{n} |\nu(E_i)| : E_i \in \mathcal{A}, \dot{\cup} E_i = E\}$.

(83) (HW4) Let $f \in L^1$, $g \in L^\infty$. Prove that $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

Practise exercises 9.

(84) (HW1) Prove that $\|f\|_\infty = \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}$ defines a norm on L^∞.

(85) (HW2) Let ν be a signed measure and μ be a measure on $(X, \mathcal{A})$. Prove that $\nu \ll \mu$ iff $|\nu| \ll \mu$.

(86) Let μ and ν_j $(j = 1, 2, \dots)$ be positive measures such that $\nu_j \ll \mu$ for all j. Prove that $\sum_{j=1}^{\infty} \nu_j \ll \mu$.

(87) (HW3) Let $F(x) = \arctan x$ if $x < 0$ and $F(x) = \frac{2x+1}{x+1}$ if $x \geq 0$. Let μ be the Lebesgue-Stieltjes measure generated by F, i.e. $\mu((a, b]) = F(b) - F(a)$. Find $\int_{-1}^{1} (1 + x) d\mu$.

(88) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, g \in L^1(\mu)$. Prove that $f = g$ a.e. iff $\int_E f = \int_E g$ for all $E \in \mathcal{A}$.

(89) Prove that if ν is a signed measure, λ, μ are positive measures such that $\nu = \lambda - \mu$ then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

(90) Prove that for any $1 < p \leq q < \infty$ we have $\ell^p \subset \ell^q$, and for any sequence $\underline{a} = (a_n)_{n \in \mathbb{N}} \in \ell^p$ we have $\|\underline{a}\|_q \leq \|\underline{a}\|_p$.

(91) (HW4) Let $p = 3$ and consider the following linear functional f on ℓ^p: for $\underline{x} = (x_1, x_2, x_3, \dots) \in \ell^p$ let $f(\underline{x}) = 4x_1 - x_2$. Prove that f is bounded and determine its norm. (Use $\ell^p - \ell^q$ duality!)

(92) Assume $1 \leq p < \infty$, $f_n, f \in L^p$ and $f_n \to f$ almost everywhere. Prove that $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$.

Practise exercises 9. (summary before retake of the Midterm)

(93) Evaluate $\int_0^1 \int_y^1 x^{-3/2} \cos \frac{\pi y}{2x} dx dy$. (Use Fubini.)

(94) Evaluate $\int_0^\infty \frac{n \sin(x/n)}{x(x^2+1)} dx$. (Use LDCT.)

(95) Find $\lim_{n\to\infty} \int_0^1 \frac{nx^n}{\log(x+2)} dx$. (Use LDCT after partial integration.)

(96) Consider the sequence of functions $f_n(x) = \sum_{k=0}^{n}(k+1)x^k$. Do they converge on the interval $[-1, 1)$
– pointwise?
– uniformly?
– in measure?
(Recall what you know about power series, and use Egoroff theorem if necessary.)

(97) Consider the sequence of functions $f_n(x) = nxe^{-nx}$. Do they converge on the half-line $[0, \infty)$
– pointwise?
– uniformly?
– in measure?
(For each fixed n analyze the behaviour of the function f_n.)

(98) Let $F(x) = \arctan(2x+1)$ if $x < 0$ and $F(x) = \frac{2x+1}{x+1}$ if $x \geq 0$. Let μ be the Lebesgue-Stieltjes measure generated by F, i.e. $\mu((a, b]) = F(b) - F(a)$. Find $\int_{-1}^{1}(2+x)d\mu$. (Decompose the measure μ_F to singular and absolutely continuous parts with respect to the Lebesgue measure.)

(99) Let $f(x) = |x|^{-\alpha}$ for $x \in \mathbb{R}^n$. For which values of $\alpha > 0$ and $1 \leq p < \infty$ do we have $f \in L^p(\mathbb{R}^n)$? (Use integration in polar coordinates.)

(100) Let $\mu(X) < \infty$, and $1 < p < q < \infty$. Prove that $L^q(X) \subset L^p(X)$, and $\|f\|_p \leq \|f\|_q(\mu(X))^{\frac{1}{p}-\frac{1}{q}}$. (Use Holder's inequality.)