

# PLÜNNECKE'S INEQUALITY FOR DIFFERENT SUMMANDS

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ABSTRACT. The aim of this paper is to prove a general version of Plünnecke's inequality. Namely, assume that for finite sets  $A, B_1, \dots, B_k$  we have information on the size of the sumsets  $A + B_{i_1} + \dots + B_{i_l}$  for all choices of indices  $i_1, \dots, i_l$ . Then we prove the existence of a non-empty subset  $X$  of  $A$  such that we have 'good control' over the size of the sumset  $X + B_1 + \dots + B_k$ . As an application of this result we generalize an inequality of [1] concerning the submultiplicativity of cardinalities of sumsets.

## 1. INTRODUCTION

Plünnecke [4] developed a graph-theoretic method to estimate the density of sumsets  $A + B$ , where  $A$  has a positive density and  $B$  is a basis. The third author published a simplified version of his proof [5, 6]. Accounts of this method can be found in Malouf [2], Nathanson [3], Tao and Vu [7].

The simplest instance of Plünnecke's inequality for finite sets goes as follows.

**Theorem 1.1.** *Let  $l < k$  be integers,  $A, B$  sets in a commutative group and write  $|A| = m$ ,  $|A + lB| = \alpha m$ . There exists an  $X \subset A$ ,  $X \neq \emptyset$  such that*

$$(1.1) \quad |X + kB| \leq \alpha^{k/l} |X|.$$

Plünnecke deduced his results from certain properties of the graph built on the sets  $A, A + B, \dots, A + kB$  as vertices (in  $k + 1$  different copies of the group), where from an  $x \in A + iB$  edges go to each  $x + b \in A + (i + 1)B$ . This property (which he called "commutativity") was based on the possibility of replacing a path from  $x$  to  $x + b + b'$  through  $x + b$  by a path through  $x + b'$ , so commutativity of addition and the fact that we add the same set  $B$  repeatedly seemed to be central ingredients of this method. Still, it is possible to relax these assumptions. Here we concentrate on the second of them.

In [5] the case  $l = 1$  of Theorem 1.1 is extended to the addition of different sets as follows.

**Theorem 1.2.** *Let  $A, B_1, \dots, B_k$  be finite sets in a commutative group and write  $|A| = m$ ,  $|A + B_i| = \alpha_i m$ , for  $1 \leq i \leq k$ . There exists an  $X \subset A$ ,  $X \neq \emptyset$  such that*

$$(1.2) \quad |X + B_1 + \dots + B_k| \leq \alpha_1 \alpha_2 \dots \alpha_k |X|.$$

The aim of this paper is to give a similar extension of the general case. This extension will then be applied in Section 6 to prove a conjecture from our paper [1].

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**Theorem 1.3.** *Let  $l < k$  be integers, and let  $A, B_1, \dots, B_k$  be finite sets in a commutative group  $G$ . Let  $K = \{1, 2, \dots, k\}$ , and for any  $I \subset K$  put*

$$B_I = \sum_{i \in I} B_i,$$

$$|A| = m, \quad |A + B_I| = \alpha_I m.$$

*(This is compatible with the previous notation if we identify a one-element subset of  $K$  with its element.) Write*

$$(1.3) \quad \beta = \left( \prod_{L \subset K, |L|=l} \alpha_L \right)^{(l-1)!(k-l)!/(k-1)!}.$$

*There exists an  $X \subset A$ ,  $X \neq \emptyset$  such that*

$$(1.4) \quad |X + B_K| \leq \beta |X|.$$

The problem of relaxing the commutativity assumption will be the subject of another paper. Here we just mention without proof the simplest case.

**Theorem 1.4.** *Let  $A, B_1, B_2$  be sets in a (typically noncommutative group)  $G$  and write  $|A| = m$ ,  $|B_1 + A| = \alpha_1 m$ ,  $|A + B_2| = \alpha_2 m$ . There is an  $X \subset A$ ,  $X \neq \emptyset$  such that*

$$(1.5) \quad |B_1 + X + B_2| \leq \alpha_1 \alpha_2 |X|.$$

The following result gives estimates for the size of the set  $X$  in Theorem 1.3 and a more general property than (1.4), but it is weaker by a constant. We do not make any effort to estimate this constant; an estimate could be derived from the proof, but we feel it is probably much weaker than the truth.

**Theorem 1.5.** *Let  $l < k$  be positive integers, and let  $A, B_1, \dots, B_k$  be finite sets in a commutative group  $G$ . Let  $K, B_I, \alpha_I$  and  $\beta$  be as in Theorem 1.3. For any  $J \subset K$  such that  $l < j = |J| \leq k$  define*

$$(1.6) \quad \beta_J = \left( \prod_{L \subset J, |L|=l} \alpha_L \right)^{(l-1)!(j-l)!/(j-1)!}.$$

*(Observe that  $\beta_K = \beta$  of (1.3).) Let furthermore a number  $\varepsilon$  be given,  $0 < \varepsilon < 1$ . There exists an  $X \subset A$ ,  $|X| > (1 - \varepsilon)m$  such that*

$$(1.7) \quad |X + B_J| \leq c \beta_J |X|$$

*for every  $J \subset K$ ,  $|J| \geq l$ . Here  $c$  is a constant that depends on  $k, l$  and  $\varepsilon$ .*

We return to the problem of finding large subsets in Section 5.

## 2. THE CASE $k = l + 1$

First we prove the case  $k = l + 1$  of Theorem 1.3 in a form which is weaker by a constant.

**Lemma 2.1.** *Let  $l$  be a positive integer,  $k = l + 1$ , and let  $A, B_1, \dots, B_k$  be finite sets in a commutative group  $G$ . Let  $K, B_I, \alpha_I$  be as in Theorem 1.3. Write*

$$\beta = \left( \prod_{L \subset K, |L|=l} \alpha_L \right)^{1/l}.$$

(Observe that this is the same as  $\beta$  of (1.3) in this particular case.) There exists an  $X \subset A$ ,  $X \neq \emptyset$  such that

$$(2.1) \quad |X + B_K| \leq c_k \beta |X|$$

with a constant  $c_k$  depending on  $k$ .

*Proof.* Let  $H_1, \dots, H_k$  be cyclic groups of order  $n_1, \dots, n_k$ , respectively, let  $H = H_1 \times H_2 \times \dots \times H_k$ , and consider the group  $G' = G \times H = G \times H_1 \times \dots \times H_k$ . Introduce the notation  $B'_i = B_i \times \{0\} \times \dots \times \{0\} \times H_i \times \{0\} \times \dots \times \{0\}$  which will be abbreviated as  $B'_i = B_i \times H_i$ , in the same manner as  $A \times \{0\} \times \dots \times \{0\}$  will still be denoted by  $A$ .

We introduce the notation  $i^* = K \setminus \{i\} = \{1, \dots, i-1, i+1, \dots, k\}$  which gives naturally  $B_{i^*} = \sum_{j \neq i} B_j$  and, correspondingly,  $\alpha_{i^*} = \alpha_{\{1, 2, \dots, i-1, i+1, \dots, k\}}$ . Note that we have  $\prod \alpha_{i^*} = \beta^l$ .

Similarly, let  $H_{i^*} = H_1 \times \dots \times H_{i-1} \times \{0\} \times H_{i+1} \times \dots \times H_k$ , and  $B'_{i^*} = \sum_{j \neq i} B'_j = B_{i^*} \times H_{i^*}$ .

Let  $q$  be a positive integer (which should be thought of as a large number), and let  $n_i = \alpha_{i^*} q$ . We restrict  $q$  to values for which these are integers; such values exist, since the numbers  $\alpha_L$  are rational. Then  $|H| = n = \prod n_i = \beta^l q^k$  and  $|H_{i^*}| = n/n_i = (\beta q)^l / \alpha_{i^*}$ . Hence  $|A + B'_{i^*}| = |A + B_{i^*}| |H_{i^*}| = m(\beta q)^l$  independently of  $i$ .

Now, let  $B' = \bigcup_{i=1}^k B'_i$ , and consider the cardinality of the set  $A + (k-1)B'$ . The point is that the main part of this cardinality comes from terms where the summands  $B'_i$  are all different, i.e. from terms of the form  $A + B'_{i^*}$ ,  $i = 1, 2, \dots, k$ . There are  $k$  such terms, so their cardinality altogether is not greater than

$$(2.2) \quad km(\beta q)^l.$$

The rest of the terms all contain some equal summands, e.g.  $A + B'_1 + B'_1 + B'_2 + B'_3 \dots + B'_{k-2}$ , containing two copies of  $B'_1$ , etc. The number of such terms is less than  $k^k$ , and each of them has 'small' cardinality for the simple reason that  $H_i + H_i = H_i$ . For instance, in the example above we have  $|A + B'_1 + B'_1 + B'_2 + B'_3 \dots + B'_{k-2}| \leq m|B_1|(\prod_{j=1}^{k-2} |B_j| n_j) \leq c(A, B_1, \dots, B_k) q^{k-2}$  where  $c(A, B_1, \dots, B_k)$  is a constant depending on the sets  $A, B_1, \dots, B_k$  but not on  $q$ . Therefore the cardinality of the terms containing some equal summands is not greater than

$$(2.3) \quad k^k c(A, B_1, \dots, B_k) q^{k-2} = c(k, A, B_1, \dots, B_k) q^{k-2} = o(q^l)$$

Therefore, combining (2.2) and (2.3) we conclude that

$$(2.4) \quad |A + (k-1)B'| \leq 2km(\beta q)^l$$

if  $q$  is chosen large enough.

Finally, we apply Theorem 1.1 to the sets  $A$  and  $B'$  in  $G'$ . We conclude by (2.4) that there exists a subset  $X \subset A$  such that

$$(2.5) \quad |X + kB'| \leq |X| (2k(\beta q)^l)^{k/l} = c_k |X| (\beta q)^k.$$

Also, observe that  $X + (B_K \times H) \subset X + kB'$ , and  $|X + (B_K \times H)| = n|X + B_K|$ . From these facts and (2.5) we obtain

$$|X + B_K| \leq c_k |X| (\beta q)^k / n = c_k \beta |X|$$

as desired.  $\square$

### 3. THE GENERAL CASE

In this section we prove Theorem 1.5.

As a first step we add a bound on  $|X|$  to Lemma 2.1.

**Lemma 3.1.** *Let  $k = l + 1$ , and let  $A, B_i, B_I, \alpha_I$  and  $\beta$  be as in Lemma 2.1. Let a number  $\varepsilon$  be given,  $0 < \varepsilon < 1$ . There exists an  $X \subset A$ ,  $|X| > (1 - \varepsilon)m$  such that*

$$(3.1) \quad |X + B_K| \leq c(k, \varepsilon) \beta |X|$$

with a constant  $c(k, \varepsilon) = c_k \varepsilon^{-\frac{k}{k-1}}$  depending on  $k$  and  $\varepsilon$ .

*Proof.* Take the largest  $X \subset A$  for which (3.1) holds. If  $|X| > (1 - \varepsilon)m$ , we are done. Assume this is not the case. Put  $A' = A \setminus X$ , and apply Lemma 2.1 with  $A'$  in the place of  $A$ . We know that  $|A'| \geq \varepsilon m$ . The assumptions will hold with

$$\alpha'_I = |A' + B_i| / |A'| \leq |A + B_i| / |A'| \leq \alpha_I / \varepsilon$$

in the place of  $\alpha_I$ . We get a nonempty  $X' \subset A'$  such that

$$|X' + B_K| \leq c_k \beta' |X'|$$

with

$$\beta' = \left( \prod_{L \subset K, |L|=l} \alpha'_L \right)^{1/(k-1)} \leq \beta \varepsilon^{-\frac{k}{k-1}}.$$

Then  $X \cup X'$  would be a larger set, a contradiction.  $\square$

Now we turn to the general case.

**Lemma 3.2.** *Let  $J_1, \dots, J_n$  be a list of all subsets of  $K$  satisfying  $l < |J| \leq k$  arranged in an increasing order of cardinality (so that  $J_n = K$ ); within a given cardinality the order of the sets may be arbitrary.*

*Let  $A, B_i, B_I, \alpha_I$  and  $\beta_I$  be as in Theorem 1.5, and let the numbers  $0 < \varepsilon < 1$  and  $1 \leq r \leq n$  be given. There exists an  $X \subset A$ ,  $|X| > (1 - \varepsilon)m$  such that*

$$(3.2) \quad |X + B_J| \leq c(k, l, r, \varepsilon) \beta_J |X|$$

for every  $J = J_1, \dots, J_r$  with a constant  $c(k, l, r, \varepsilon)$  depending on  $k, l, r$  and  $\varepsilon$ .

Theorem 1.5 is the case  $r = n$ .

*Proof.* We shall prove the statement by induction on  $r$ . Since the sets are in increasing order of size, we have  $|J_1| = l + 1$ , and the claim for  $r = 1$  follows from Lemma 3.1.

Now assume we know the statement for  $r - 1$ . We apply it with  $\varepsilon/2$  in the place of  $\varepsilon$ , so we have a set  $X \subset A$ ,  $|X| > (1 - \varepsilon/2)m$  such that (3.2) holds for  $J = J_1, \dots, J_{r-1}$  with  $c(k, l, r - 1, \varepsilon/2)$ . Write  $A' = X$ . This set satisfies the assumptions with

$$\alpha'_I = \alpha_I / (1 - \varepsilon/2).$$

We have  $|J_r| = k'$  with some  $k'$ ,  $l < k' \leq k$ . We are going to apply Lemma 3.1 with  $A', k'$  in the place of  $A, k$  and  $\varepsilon/2$  in the place of  $\varepsilon$ . To this end we need bounds for  $|A' + B_L|$  for every  $L$  such that  $|L| = l' = k' - 1$ . By the inductive assumption we know

$$|A' + B_L| \leq c(k, l, r - 1, \varepsilon/2)\beta_L |A'|.$$

Lemma 3.1 gives us a set  $X' \subset A'$  such that

$$|X'| > (1 - \varepsilon/2) |A'| > (1 - \varepsilon)m$$

and

$$|X' + B_{J_r}| \leq c(l', \varepsilon/2)\beta' |X'|,$$

where

$$\beta' = \left( \prod_{L \subset J_r, |L|=l'} c(k, l, r - 1, \varepsilon/2)\beta_L \right)^{1/l} = c(k, l, r - 1, \varepsilon/2)\beta_{J_r}.$$

In the last step we used an identity among the quantities  $\beta_J$  which easily follows from their definition (1.6).

The desired set  $X$  will be this  $X'$ , and the value of the constant is

$$c(k, l, r, \varepsilon) = c(l', \varepsilon/2)c(k, l, r - 1, \varepsilon/2).$$

□

#### 4. REMOVING THE CONSTANT

In this section we prove Theorem 1.3. This is done with the help of Theorem 1.5 and technique of taking direct powers of the appearing groups, sets, and corresponding graphs.

*Proof of Theorem 1.3.* Consider the following bipartite directed graph  $\mathcal{G}^1$ . The first collection of vertices  $V_1$  are the elements of set  $A$ , and the second collection of vertices  $V_2$  are the elements of set  $A + B_K$  (taken in two different copies of the ambient group to make them disjoint). There is an edge in  $\mathcal{G}^1$  from  $v_1 = a_1 \in V_1$  to  $v_2 = a_2 + b_{1,2} + \dots + b_{k,2} \in V_2$  if and only if there exist elements  $b_{1,1}, \dots, b_{k,1}$  such that  $a_1 + b_{1,1} + \dots + b_{k,1} = a_2 + b_{1,2} + \dots + b_{k,2}$ . The image of a set  $Z \subset V_1$  is the set  $\text{im } Z \subset V_2$  reachable from  $Z$  via edges. The *magnification ratio*  $\gamma$  of the the graph  $\mathcal{G}^1$  is  $\min\{\frac{|\text{im } Z|}{|Z|}, Z \subset V_1\}$ . The statement of Theorem 1.3 in these terms is that  $\gamma \leq \beta$ , with  $\beta$  as defined in the theorem.

Consider now the direct power  $\mathcal{G}^r = \mathcal{G}^1 \times \mathcal{G}^1 \times \dots \times \mathcal{G}^1$  with collections of edges  $V_1^r = V_1 \times \dots \times V_1$  and  $V_2^r = V_2 \times \dots \times V_2$ , and edges from  $(v_1^1, v_2^1, \dots, v_r^1) \in V_1^r$  to  $(v_1^2, v_2^2, \dots, v_r^2) \in V_2^r$  if and only if there exist  $\mathcal{G}^1$ -edges in each of the coordinates. Observe that the directed graph  $\mathcal{G}^r$  corresponds exactly to the sets  $A^r$  and  $A^r + (B_1^r + \dots + B_k^r)$  in the direct power group  $G^r$ . Applying Theorem 1.5 in the group  $G^r$  to the sets  $A^r, B_1^r, \dots, B_k^r$  with any fixed  $\varepsilon$ , say  $\varepsilon = 1/2$ , we obtain that the magnification ration  $\gamma_r$  of  $\mathcal{G}^r$  is not greater than  $c\beta^r$ . On the other hand, the magnification ratio is multiplicative (see [5] or [3]), so that we have  $\gamma_r = \gamma^r$ . Therefore we conclude that  $\gamma \leq \sqrt[r]{c}\beta$  and, in the limit,  $\gamma \leq \beta$  as desired. □

## 5. FINDING A LARGE SUBSET

We give an effective version of Theorem 1.5 in the original case, that is, when only  $X + B_K$  needs to be small.

**Theorem 5.1.** *Let  $A, B_i, B_I, \alpha_I$  and  $\beta$  be as in Theorem 1.3.*

(a) *Let an integer  $a$  be given,  $1 \leq a \leq m$ . There exists an  $X \subset A$ ,  $|X| \geq a$  such that*

$$(5.1) \quad |X + B_K| \leq \beta m^{k/l} (m^{-k/l} + (m-1)^{-k/l} + \cdots + (m-a+1)^{-k/l} + (|X| - a)(m-a+1)^{-k/l}).$$

(b) *Let a real number  $t$  be given,  $0 \leq t < m$ . There exists an  $X \subset A$ ,  $|X| > t$  such that*

$$(5.2) \quad |X + B_K| \leq \beta m^{k/l} \left( \frac{l}{k-l} ((m-t)^{1-k/l} - m^{1-k/l}) + (|X| - t)(m-t)^{-k/l} \right).$$

*Proof.* To prove (a), we use induction on  $a$ . The case  $a = 1$  is Theorem 1.3. Now suppose we know it for  $a$ ; we prove it for  $a + 1$ . The assumption gives us a set  $X$ ,  $|X| \geq a$  with a bound on  $|X + B_K|$  as given by (5.1). We want to find a set  $X'$  with  $|X'| \geq a + 1$  and

$$(5.3) \quad |X' + B_K| \leq \beta m^{k/l} (m^{-k/l} + (m-1)^{-k/l} + \cdots + (m-a)^{-k/l} + (|X| - a - 1)(m-a)^{-k/l}).$$

If  $|X| \geq a + 1$ , we can put  $X' = X$ . If  $|X| = a$ , we apply Theorem 1.3 to the sets  $A' = A \setminus X, B_1, \dots, B_k$ . In doing this the numbers  $\alpha_I$  should be replaced by

$$\alpha'_I = \frac{|A' + B_I|}{|A'|} \leq \frac{|A + B_I|}{|A'|} = \alpha_I \frac{m}{m-a}.$$

This yields a set  $Y \subset A \setminus X$  such that

$$|Y + B_K| \leq \beta' |Y|$$

with

$$\beta' = \left( \prod_{L \subset K, |L|=l} \alpha'_L \right)^{(l-1)!(k-l)!/(k-1)!} \leq \beta \left( \frac{m}{m-a} \right)^{k/l}$$

and we put  $X' = X \cup Y$ .

To prove part (b) we apply (5.1) with  $a = [t] + 1$ . The right side of (5.2) can be written as  $\beta m^{k/l} \int_0^{|X|} f(x) dx$ , where  $f(x) = (m-x)^{-k/l}$  for  $0 \leq x \leq t$ , and  $f(x) = (m-t)^{-k/l}$  for  $t < x \leq |X|$ . Since  $f$  is increasing, the integral is  $\geq f(0) + f(1) + \cdots + f(|X| - 1)$ . This exceeds the right side of (5.1) by a termwise comparison.  $\square$

## 6. AN APPLICATION TO RESTRICTED SUMS

We prove the following result, which was conjectured in [1].

**Theorem 6.1.** *Let  $A, B_1, \dots, B_k$  be finite sets in a commutative group, and  $S \subset B_1 + \cdots + B_k$ . We have*

$$(6.1) \quad |S + A|^k \leq |S| \prod_{i=1}^k |A + B_1 + \cdots + B_{i-1} + B_{i+1} + \cdots + B_k|.$$

Two particular cases were established in [1]; the case when  $S$  is the complete sum  $B_1 + \cdots + B_k$ , and the case  $k = 2$ . The proof in the sequel is similar to the proof of the case  $k = 2$ , the main difference being that we use the above generalized Plünnecke inequality, while for  $k = 2$  the original was sufficient.

*Proof.* Let us use the notation  $|A| = m$ ,  $s = \prod_{i=1}^k |A + B_1 + \cdots + B_{i-1} + B_{i+1} + \cdots + B_k|$ . Observe that if  $|S| \leq (s/m^k)^{\frac{1}{k-1}}$  then

$$(6.2) \quad |S + A| \leq |S||A| = |S|^{\frac{1}{k}} |S|^{\frac{k-1}{k}} m \leq (|S|s)^{\frac{1}{k}}$$

and we are done.

If  $|S| > (s/m^k)^{\frac{1}{k-1}}$ , then we will use Theorem 5.1, part (b) with  $l = k - 1$ . Note that the  $\beta$  of this theorem can be expressed by our  $s$  as

$$\beta = s^{1/(k-1)} m^{-k/(k-1)}.$$

We take  $t = m - \left(\frac{s}{|S|^{k-1}}\right)^{1/k}$ . Then there exists a set  $X \subset A$  such that  $|X| = r > t$  and (5.2) holds. For such an  $X$  we have

$$(6.3) \quad |S+X| \leq |B_K+X| \leq (k-1)s^{\frac{1}{k-1}} \left( (m-t)^{-\frac{1}{k-1}} - m^{-\frac{1}{k-1}} \right) + (r-t) \left( \frac{s}{(m-t)^k} \right)^{\frac{1}{k-1}}$$

and we add to this the trivial bound

$$(6.4) \quad |S + (A \setminus X)| \leq |S||A \setminus X| = |S|(m - r).$$

We conclude that

$$(6.5) \quad |S + A| \leq |S + X| + |S + (A \setminus X)| \leq (k-1)s^{\frac{1}{k-1}} \left( (m-t)^{-\frac{1}{k-1}} - m^{-\frac{1}{k-1}} \right) +$$

$$(r-t) \left( \frac{s}{(m-t)^k} \right)^{\frac{1}{k-1}} + |S|((m-t) - (r-t)) = ks^{1/k} |S|^{1/k} - (k-1) \left( \frac{s}{m} \right)^{\frac{1}{k-1}} \leq k(s|S|)^{1/k}$$

This inequality is nearly the required one, except for the factor  $k$  on the right hand side. We can dispose of this factor as follows (once again, the method of direct powers). Consider the sets  $A' = A^r$ ,  $B'_j = B_j^r$  ( $j = 1, \dots, k$ ), and  $S' = S^r$  in the  $r$ 'th direct power of the original group. Applying equation (6.5) to  $A'$ , etc., we obtain

$$(6.6) \quad |S' + A'| \leq k(s'|S'|)^{1/k}.$$

Since  $|S' + A'| = |S + A|^r$ ,  $s' = s^r$  and  $|S'| = |S|^r$ , we get

$$(6.7) \quad |S + A| \leq k^{1/r} (s|S|)^{1/k}.$$

Taking the limit as  $r \rightarrow \infty$  we obtain the desired inequality

$$(6.8) \quad |S + A| \leq (s|S|)^{1/k}.$$

□

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