

A NOTE ON THE PYJAMA PROBLEM

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ABSTRACT. This note concerns the so-called pyjama problem, whether it is possible to cover the plane by finitely many rotations of vertical strips of half-width ε . We first prove that there exist no periodic coverings for $\varepsilon < \frac{1}{3}$. Then we describe an explicit (non-periodic) construction for $\varepsilon = \frac{1}{3} - \frac{1}{48}$. Finally, we use a compactness argument combined with some ideas from additive combinatorics to show that a finite covering exists for $\varepsilon = \frac{1}{5}$. The question whether ε can be arbitrarily small remains open.

1. INTRODUCTION

This note concerns a question that has been advertised as the "pyjama-problem" in the additive combinatorics community. The problem was originally raised in [5], and we recall it here for convenience. Let $\|x\|$ denote the distance of any real number x to the closest integer, and define the following set of equidistant vertical strips of width 2ε on \mathbb{R}^2 :

$$E_\varepsilon := \{(x, y) \in \mathbb{R}^2 : \|x\| \leq \varepsilon\}.$$

Denote by R_θ the counterclockwise rotation of the plane by angle θ (around the origin). The question is whether we can cover the plane by the union of finitely many rotates of E_ε , i.e. whether there exist angles $\theta_0, \dots, \theta_n$ such that $\mathbb{R}^2 = \bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$. We will assume throughout this note (without loss of generality) that the angles $\theta_0, \dots, \theta_n$ are pairwise distinct.

We make a few remarks on the origin of the problem. In [3] Furstenberg, Katznelson and Weiss proved that for any set A of positive upper density in \mathbb{R}^2 there exists a threshold $t_0 \in \mathbb{R}$ such that for any $t \geq t_0$ there exist points in A with distance t . Another proof was given by Falconer and Marstrand in [2]. Subsequently, Bourgain [1] used a Fourier analytic argument to generalize the result in higher dimensions: a set A of positive upper density in \mathbb{R}^k contains all large enough copies of any

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$k - 1$ dimensional simplex. Kolountzakis [4] used a similar argument to prove that even if the unit ball is non-Euclidean (but smooth), all large enough distances appear between points of A . In this circle of problems, a natural question of Sz. Révész (private communication) was whether it is true in \mathbb{R}^2 that finitely many *rotations* of the difference set $A - A$ can cover the plane, with the exception of a bounded set. This would have given a generalization of the above mentioned result of [2, 3]. However, the question was answered in the negative in [5], where A was taken to be a set of small disks around points of the integer lattice. The "pyjama-problem" was formulated and left open in [5]. For further related problems concerning extremal properties of sets of positive upper density we refer to [6].

Let us introduce some notations and definitions.

Definition 1.1. *We will say that ε has the finite rotation property if there exist angles $\theta_0, \dots, \theta_n$ such that $\mathbb{R}^2 = \bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$. Let ε_0 denote the infimum of the values of ε having the finite rotation property.*

We believe that ε can be arbitrarily small.

Conjecture 1.1. *With notation introduced above, we have $\varepsilon_0 = 0$.*

Let $\mathbf{u}_j = (\cos \theta_j, \sin \theta_j) \in \mathbb{R}^2$ be the unit vector corresponding to the angle θ_j . It is easy to see that a vector $\mathbf{x} \in \mathbb{R}^2$ is covered by $R_{\theta_j} E_\varepsilon$ if and only if $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| \leq \varepsilon$. The pyjama problem can therefore be formulated in an equivalent way as follows:

For a given $\varepsilon > 0$ we want to find unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{R}^2$ such that for all $\mathbf{x} \in \mathbb{R}^2$ there exists a \mathbf{u}_j such that $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| \leq \varepsilon$.

The case $\varepsilon = \frac{1}{3}$ is "trivial". Indeed, the rotations by angles $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ will suffice, as the reader can easily verify. As we shall see, it is not at all trivial to go below $\varepsilon = \frac{1}{3}$. The above covering by rotations $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ is *periodic*. One natural approach is to consider other periodic arrangements of the strips (e.g. angles corresponding to Pythagorean triples). We will make the concept of periodicity rigorous in Section 2, and prove that it can never work for any $\varepsilon < \frac{1}{3}$.

Another natural approach is to consider angles corresponding to N th roots of unity for some N . It can be proven, however, that we will not get a covering of \mathbb{R}^2 in this manner for any N and any $\varepsilon < \frac{1}{3}$. We will not include the proof of this negative result to keep this note brief.

A random set of angles will not lead to a covering for any $\varepsilon < \frac{1}{2}$, almost surely. The reason is that the numbers $\cos \theta_0, \dots, \cos \theta_n$ will be almost surely independent over \mathbb{Q} , and therefore the set $\mathbb{R}(\cos \theta_0, \dots, \cos \theta_n)$,

mod 1, will be dense in the torus \mathbb{T}^{n+1} , and thus we can find a vector $\mathbf{x} = (x, 0) \in \mathbb{R}^2$ such that $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| \approx \frac{1}{2}$ for all j . A similar argument shows that we will not get a covering for any $\varepsilon < \frac{1}{2}$ if the vectors $\mathbf{u}_0, \dots, \mathbf{u}_n$ are independent over \mathbb{Q} (in that case one needs to consider $\mathbf{x} = (x \cos \varphi, x \sin \varphi)$ for some appropriate angle φ and $x \in \mathbb{R}$).

In Section 3 we will show a specific finite set of angles for $\varepsilon = \frac{1}{3} - \frac{1}{48}$. Finally, in Section 4 we will use a compactness argument combined with some ideas from additive combinatorics to show that a finite covering exists for $\varepsilon = \frac{1}{5}$. However, this result is *non-constructive*, i.e. we will not be able to exhibit the appropriate angles.

2. PERIODIC COVERING IS NOT POSSIBLE FOR $\varepsilon < \frac{1}{3}$

Periodicity is a natural idea to ensure that the whole plane gets covered by the rotated strips. The reason is that in this case only the fundamental region (which is a finite parallelogram, spanned by the period-vectors) needs to be checked. However, we will now show that ε cannot be smaller than $\frac{1}{3}$ in such a case.

Definition 2.1. *Given $\theta_0, \dots, \theta_n$, and the corresponding unit vectors $\mathbf{u}_0, \dots, \mathbf{u}_n$, a vector $\mathbf{v} \in \mathbb{R}^2$ is a period-vector of the set $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$, if $\langle \mathbf{u}_j, \mathbf{v} \rangle \in \mathbb{Z}$ for each j . The set $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$ is called (fully) periodic if it has two \mathbb{R} -linearly independent period vectors $\mathbf{v}_0, \mathbf{v}_1$.*

Periodicity is directly related to the dimension of the space spanned by the vectors $\mathbf{u}_0, \dots, \mathbf{u}_n$ over \mathbb{Q} .

Lemma 2.1. *For any $n \geq 1$ the set $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$ is periodic if and only if $d := \dim(\text{span}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}_{\mathbb{Q}}) = 2$.*

Proof. Assume $d = 2$. As \mathbf{u}_0 and \mathbf{u}_1 are distinct, we can find two \mathbb{R} -linearly independent vectors $\mathbf{v}_0, \mathbf{v}_1$ such that $\langle \mathbf{u}_j, \mathbf{v}_k \rangle \in \mathbb{Z}$ for $0 \leq j, k \leq 1$. Let $\mathbf{u}_2 = q_0 \mathbf{u}_0 + q_1 \mathbf{u}_1$ where $q_0, q_1 \in \mathbb{Q}$, and let M denote the least common multiple of the denominators of q_0, q_1 . Then it is straightforward to check that the vectors $\mathbf{w}_0 = M \mathbf{v}_0$, $\mathbf{w}_1 = M \mathbf{v}_1$ are two period-vectors with respect to $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$. We can then proceed by induction to produce two period vectors with respect to $\mathbf{u}_0, \dots, \mathbf{u}_n$.

Assume now that $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$ is periodic, i.e. there exist two linearly independent period vectors $\mathbf{v}_0, \mathbf{v}_1$ such that $\langle \mathbf{u}_j, \mathbf{v}_k \rangle := m_{j,k} \in \mathbb{Z}$ for each $j = 0, \dots, n$ and $k = 0, 1$. We need to prove that each \mathbf{u}_j is a \mathbb{Q} -linear combination of \mathbf{u}_0 and \mathbf{u}_1 . Let us fix j . The 2×2 matrix \mathbf{A} given by $a_{k,r} = m_{r,k}$ (for $0 \leq k, r \leq 1$) contains integer entries and is non-singular. Therefore, there exists a vector $\mathbf{q}_j = \begin{pmatrix} q_{0,j} \\ q_{1,j} \end{pmatrix}$ with

rational coordinates such that $\mathbf{A} \cdot \mathbf{q}_j = \begin{pmatrix} -m_j, 0 \\ -m_j, 1 \end{pmatrix}$. But then the vector $\mathbf{u}_j + q_{1,j}\mathbf{u}_1 + q_{0,j}\mathbf{u}_0$ is orthogonal to both \mathbf{v}_0 and \mathbf{v}_1 , and therefore must be zero. \square

Lemma 2.2. *If $n \geq 2$, $\theta_0 = 0$ and $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$ is periodic, then all $e^{i\theta_j}$ (for $j = 0, \dots, n$) belong to the same quadratic imaginary field.*

Proof. As we saw in the previous lemma, there are non-zero rational numbers q_0, q_1 , such that $q_0 + q_1 e^{i\theta_1} = e^{i\theta_2}$. Hence $|q_0 + q_1 e^{i\theta_1}|^2 = 1$, which yields that $\cos \theta_1 = \frac{1 - q_0^2 - q_1^2}{2q_0 q_1}$ is rational (and similarly, all $\cos \theta_j$ are rational). Furthermore, $e^{i\theta_1}$ satisfies the quadratic equation $x^2 - 2 \cos \theta_1 x + 1 = 0$. As all $e^{i\theta_j}$ are \mathbb{Q} -linear combinations of $e^{i\theta_0} = 1$ and $e^{i\theta_1}$, they all belong to the same imaginary quadratic field $\mathbb{Q}(e^{i\theta_1})$. \square

We are now in position to prove the main (negative) result of this section.

Theorem 2.3. *If $\varepsilon < \frac{1}{3}$ and $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$ is periodic, then it does not cover the whole plane.*

Proof. If $n = 0$ or 1 then it is trivial that covering is not possible for any $\varepsilon < \frac{1}{2}$.

Assume $n \geq 2$, and $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$ is periodic. Without loss of generality we can assume that $\theta_0 = 0$. Then all the numbers $e^{i\theta_j}$ (for $j = 0, \dots, n$) belong to the same quadratic imaginary field, and all the numbers $\cos \theta_j$ are rational by Lemma 2.2. Let the field be denoted by $\mathbb{K} = \mathbb{Q}(\sqrt{-D})$, where D is a positive square-free integer. Then $(\cos \theta_j, \sin \theta_j) = (\frac{m_j}{n_j}, \frac{k_j \sqrt{D}}{n_j})$ for some integers m_j, k_j, n_j such that $m_j^2 + Dk_j^2 = n_j^2$. We may assume that $\gcd(m_j, k_j, n_j) = 1$; then the equation $m_j^2 + Dk_j^2 = n_j^2$ implies that m_j, k_j, n_j are pairwise co-prime. Let $M = \prod_{j=0}^n n_j$.

If $D = 1$ then the angles θ_j correspond to Pythagorean triples, $m_j^2 + k_j^2 = n_j^2$. Note that all the n_j must be odd. Consider the point $\mathbf{x} = (\frac{M}{2}, \frac{M}{2}) \in \mathbb{R}^2$. Then $\langle \mathbf{u}_j, \mathbf{x} \rangle = \frac{M}{n_j} \left(\frac{m_j + k_j}{2} \right)$, where $\frac{M}{n_j}$ is an odd integer and $m_j + k_j$ is odd. Therefore, $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| = \frac{1}{2}$ for each j , and hence \mathbf{x} is not covered if $\varepsilon < \frac{1}{2}$.

If $D \neq 1$ then let p denote the smallest prime dividing D . If $p = 2$ then all m_j, n_j must be odd. Consider the point $\mathbf{x} = (\frac{M}{2}, M\sqrt{2}) \in \mathbb{R}^2$. Then $\langle \mathbf{u}_j, \mathbf{x} \rangle = \frac{M}{n_j} \frac{m_j}{2} + \frac{2Mk_j}{n_j}$. Therefore $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| = \frac{1}{2}$ for each j , and hence \mathbf{x} is not covered if $\varepsilon < \frac{1}{2}$.

Finally, assume $p > 2$. Then $m_j^2 + Dk_j^2 = n_j^2$ implies that $m_j \equiv \pm n_j \pmod{p}$, and all m_j, n_j are relatively prime to p . Let t be an integer such that $tM \equiv 1 \pmod{p}$. Consider the point $\mathbf{x} = (\frac{tM(p-1)}{2p}, M\sqrt{D}) \in \mathbb{R}^2$. Then $\langle \mathbf{u}_j, \mathbf{x} \rangle = \frac{tMm_j}{n_j} \frac{p-1}{2p} + \frac{k_jMD}{n_j}$. Here $\frac{tMm_j}{n_j}$ is an integer which is $\pm 1 \pmod{p}$, and $\frac{k_jMD}{n_j}$ is also an integer. Therefore, $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| = \frac{p-1}{2p} \geq \frac{1}{3}$ for each j , and hence \mathbf{x} is not covered if $\varepsilon < \frac{1}{3}$. \square

3. COVERING WITH $\varepsilon = \frac{1}{3} - \frac{1}{48}$

Having all the negative results so far, one might be tempted to conjecture that the trivial covering cannot be improved and $\varepsilon_0 = \frac{1}{3}$. However, we will now show that this is not the case.

Theorem 3.1. *Let $\varepsilon = \frac{1}{3} - \frac{1}{48}$. Define $\theta_1 = 0$, and $\theta_2 = \frac{2\pi}{3}, \theta_3 = \frac{4\pi}{3}$. Let θ_4 be such that $(\cos \theta_4, \sin \theta_4) = (\frac{1}{3}, \frac{\sqrt{8}}{3})$, and $\theta_5 = \theta_4 + \frac{2\pi}{3}, \theta_6 = \theta_4 + \frac{4\pi}{3}$. Let θ_7 be such that $(\cos \theta_7, \sin \theta_7) = (\frac{1}{3}, -\frac{\sqrt{8}}{3})$, and $\theta_8 = \theta_7 + \frac{2\pi}{3}, \theta_9 = \theta_7 + \frac{4\pi}{3}$. Then $\bigcup_{j=1}^9 R_{\theta_j} E_\varepsilon = \mathbb{R}^2$.*

Proof. Let $\mathbf{x} \in \mathbb{R}^2$ be arbitrary, and let $\langle \mathbf{u}_j, \mathbf{x} \rangle = s_j$ for $1 \leq j \leq 9$. Observe the following relations:

$$s_1 + s_2 + s_3 = 0, \quad s_4 + s_5 + s_6 = 0, \quad s_7 + s_8 + s_9 = 0, \quad 3(s_4 + s_7) - 2s_1 = 0.$$

Let \mathbf{A} denote the 4×9 matrix corresponding to this set of linear equations. Assume, by contradiction, that $\|s_j\| > \varepsilon$ for all j . Then the fractional parts $w_j = \{s_j\}$ must lie in the interval $I_\varepsilon = (\frac{1}{3} - \frac{1}{48}, \frac{2}{3} + \frac{1}{48})$. Therefore the vector $\mathbf{w} = (w_1, \dots, w_9)$ is contained in the cube I_ε^9 , and the image of \mathbf{w} under the linear transformation \mathbf{A} must be an integer lattice point in \mathbb{R}^4 . We will show that this is not possible.

We claim that all the w_j must fall into $I_1 \cup I_2$, where $I_1 = (\frac{1}{3} - \frac{1}{48}, \frac{1}{3} + \frac{1}{24})$ and $I_2 = (\frac{2}{3} - \frac{1}{24}, \frac{2}{3} + \frac{1}{48})$. Indeed, $w_1 \equiv -w_2 - w_3 \pmod{1}$, and $-w_2 - w_3 \in -I_\varepsilon - I_\varepsilon = (-\frac{4}{3} - \frac{1}{24}, -\frac{2}{3} + \frac{1}{24}) \equiv [0, \frac{1}{3} + \frac{1}{24}) \cup (\frac{2}{3} - \frac{1}{24}, 1) \pmod{1}$, and hence w_1 must fall into the intersection of this set with I_ε which is exactly $I_1 \cup I_2$. The same reasoning works for all w_j .

Finally, if all the w_j fall into $I_1 \cup I_2$, then the equation $3(w_4 + w_7) - 2w_1 \equiv 0 \pmod{1}$ cannot be satisfied. The reason for this is that $\|3(w_4 + w_7)\| < \frac{1}{4}$ (because $w_4, w_7 \in I_1 \cup I_2$), while $\|-2w_1\| > \frac{1}{4}$ (because $w_1 \in I_1 \cup I_2$). \square

This construction can be improved to decrease the value of ε . However, we do not see any argument to show that ε can be arbitrarily close to zero.

4. A COMPACTNESS ARGUMENT FOR $\varepsilon = \frac{1}{5}$

We now turn to a non-constructive compactness argument which allows us to decrease the value of ε .

Lemma 4.1. *Let \mathbb{T} denote the group $[-\frac{1}{2}, \frac{1}{2})$ with the addition operation mod 1. If ε does not have the finite rotation property then there exists a non-continuous additive homomorphism $\gamma : \mathbb{R}^2 \rightarrow \mathbb{T}$ such that $|\gamma(\mathbf{u})| \geq \varepsilon$ for all unit vectors \mathbf{u} . Conversely, if ε has the finite rotation property then there exists no additive homomorphism $\gamma : \mathbb{R}^2 \rightarrow \mathbb{T}$ such that $|\gamma(\mathbf{u})| > \varepsilon$ for all unit vectors \mathbf{u} .*

Proof. Let Γ denote the dual group of \mathbb{R}^2 (\mathbb{R}^2 is meant here as an additive group with the Euclidean topology). Then Γ can be identified with \mathbb{R}^2 in the usual way, $\mathbf{x} \leftrightarrow \gamma_{\mathbf{x}}$ where $\gamma_{\mathbf{x}} : \mathbb{R}^2 \rightarrow \mathbb{T}$ is the character $\gamma_{\mathbf{x}}(\mathbf{u}) := \langle \mathbf{x}, \mathbf{u} \rangle \pmod{1}$. Now, consider \mathbb{R}^2 as an additive group with the discrete topology. Then its dual group, denoted by Γ' , is compact and consists of all possible additive homomorphisms from $\mathbb{R}^2 \rightarrow \mathbb{T}$.

Let \mathbb{C}_1 denote the unit circle in the plane \mathbb{R}^2 . The assumption that ε does not have the finite rotation property means that for any $\mathbf{u}_1, \dots, \mathbf{u}_N \in \mathbb{C}_1$ there exists an $\mathbf{x} \in \mathbb{R}^2$ such that $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| \geq \varepsilon$ for each \mathbf{u}_j . In other words, there exists a $\gamma_{\mathbf{x}} \in \Gamma \subset \Gamma'$ such that $|\gamma_{\mathbf{x}}(\mathbf{u}_j)| \geq \varepsilon$ for each \mathbf{u}_j . Now, due to the compactness of Γ' we claim that there must exist $\gamma \in \Gamma'$ such that $|\gamma| \geq \varepsilon$ on the whole of \mathbb{C}_1 . Indeed, this is the so-called finite intersection property of compact sets: if $F_{\mathbf{u}}$ denotes the set of characters $\gamma \in \Gamma'$ such that $|\gamma(\mathbf{u})| \geq \varepsilon$ then our condition says that any finite intersection of such sets $F_{\mathbf{u}}$ is non-empty. Note that $F_{\mathbf{u}}$ are closed sets, and therefore the intersection of all of the sets $F_{\mathbf{u}}$ is non-empty by compactness.

We now prove the converse statement. If ε has the finite rotation property then there exist unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ such that for every $\mathbf{x} \in \mathbb{R}^2$ we have $\|\langle \mathbf{u}_j, \mathbf{x} \rangle\| \leq \varepsilon$ for some j . Let $M \subset \mathbb{Z}^n$ describe the rational linear relations among the vectors: $M = \{(m_1, \dots, m_n) \in \mathbb{Z}^n : \sum_j m_j \mathbf{u}_j = 0\}$. Let $g : \mathbb{R}^2 \rightarrow \mathbb{T}^n$ be the function defined by $g(\mathbf{x}) = (\langle \mathbf{u}_1, \mathbf{x} \rangle, \dots, \langle \mathbf{u}_n, \mathbf{x} \rangle)$, and let $S = \overline{\text{Ran}(g)} \subset \mathbb{T}^n$ denote the closure of the range of g . Then S is a closed subgroup, and $S \cap (\varepsilon, 1 - \varepsilon)^n = \emptyset$. The subgroup S is characterized by the linear relations in M , namely $S = \{(x_1, \dots, x_n) \in \mathbb{T}^n : \sum_j m_j x_j = 0 \text{ for all } (m_1, \dots, m_n) \in M\}$.

Consider any additive homomorphism $\gamma : \mathbb{R}^2 \rightarrow \mathbb{T}$. For every $\mathbf{x} \in \mathbb{R}^2$ we have $(\gamma(\mathbf{u}_1), \dots, \gamma(\mathbf{u}_n)) \in S$ because $\sum_j m_j \gamma(\mathbf{u}_j) = \gamma(\sum_j m_j \mathbf{u}_j) = 0$. As $S \cap (\varepsilon, 1 - \varepsilon)^n = \emptyset$ we conclude that $|\gamma(\mathbf{u}_j)| \leq \varepsilon$ for some j . \square

In the following auxiliary result it will be convenient to identify \mathbb{R}^2 with \mathbb{C} .

Lemma 4.2. *Let $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{C}$ be unit vectors, and let $M \subset \mathbb{Z}^n$ describe the rational linear relations among them: $M = \{(m_1, \dots, m_n) \in \mathbb{Z}^n : \sum_j m_j \mathbf{u}_j = 0\}$. Let $S \subset \mathbb{T}^n$ be the subgroup $S = \{(x_1, \dots, x_n) \in \mathbb{T}^n : \sum_j m_j x_j = 0 \text{ for all } (m_1, \dots, m_n) \in M\}$. Let $\gamma : \mathbb{C} \rightarrow \mathbb{T}$ be a non-continuous additive homomorphism, and let $g : \mathbb{C} \rightarrow \mathbb{T}^n$ be the function defined by $g(z) = (\gamma(\mathbf{u}_1 z), \dots, \gamma(\mathbf{u}_n z))$. Let U be any neighbourhood of zero. Then $g(U)$ is dense in S .*

Proof. We use the standard notation \overline{A} for the closure of a set A . We will show that as U runs through the neighbourhoods of zero, we have $B := \bigcap_U \overline{g(U)} = S$ (this is clearly equivalent to the statement of the proposition). First notice that B is a non-empty ($0 \in B$) and compact set. It is also clear that $B \subset S$, as $g(z) \in S$ for every $z \in \mathbb{C}$. We claim that B is a subgroup of S . To see this, let $b_1, b_2 \in B$, and let U be any neighbourhood of zero. We want to show that $b_1 - b_2 \in \overline{g(U)}$. Let U' be a smaller neighbourhood, such that $U' - U' \subset U$. Then $b_1 - b_2 \in \overline{g(U')} - \overline{g(U')} = \overline{g(U') - g(U')} = \overline{g(U' - U')} \subset \overline{g(U)}$. Therefore B is a subgroup.

If $B = S$ then we are done. If B is a compact non-empty proper subgroup of S , then there exists a continuous character $\chi : \mathbb{T}^n \rightarrow \mathbb{T}$ such that χ is not identically zero on S , but $\chi|_B \equiv 0$. Such a character χ can be identified with an n -tuple of integers, $\chi = (a_1, \dots, a_n) \in \mathbb{Z}^n$, $\chi \notin M$, so that $\chi(t_1, \dots, t_n) = a_1 t_1 + \dots + a_n t_n$ for any $(t_1, \dots, t_n) \in \mathbb{T}^n$. Let $\beta = \sum_j a_j \mathbf{u}_j$, which is non-zero because $(a_1, \dots, a_n) \notin M$. Consider the additive homomorphism $h : \mathbb{C} \rightarrow \mathbb{T}$ defined by $h(z) = \gamma(\beta z) = \chi(g(z))$. We claim that $h(z)$ is continuous. Assume it is not. Then there exists a sequence $z_m \rightarrow 0$ such that $h(z_m)$ does not converge to 0. By passing to a subsequence, we may assume that $h(z_m) \rightarrow w \neq 0$. Again, by passing to a subsequence we may assume that $g(z_m)$ converges to some $\mathbf{y} \in \mathbb{T}^n$. But then $\mathbf{y} \in B$ and $\chi(\mathbf{y}) = \lim \chi(g(z_m)) = \lim h(z_m) = w \neq 0$, a contradiction. Therefore, $h(z)$ is continuous, and so is $\gamma(z) = h(\frac{1}{\beta} z)$, which contradicts our assumption on γ . \square

With the help of Lemma 4.1 and 4.2 we can prove the main result of this section: even if $\bigcup_{j=0}^n R_{\theta_j} E_\varepsilon$ does not cover \mathbb{R}^2 but the non-covered region is small enough in some sense, we can still conclude that $\varepsilon_0 \leq \varepsilon$.

Theorem 4.3. *Assume $\varepsilon > 0$ and unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{C}$ are given. Let $M \subset \mathbb{Z}^n$ describe the rational linear relations among the vectors: $M = \{(m_1, \dots, m_n) \in \mathbb{Z}^n : \sum_j m_j \mathbf{u}_j = 0\}$. Let $S \subset \mathbb{T}^n$ be the subgroup $S = \{(x_1, \dots, x_n) \in \mathbb{T}^n : \sum_j m_j x_j = 0 \text{ for all } (m_1, \dots, m_n) \in M\}$, and let $X_\varepsilon = S \cap (\varepsilon, 1 - \varepsilon)^n$. If the difference set $X_\varepsilon - X_\varepsilon$ is not dense in S then $\varepsilon_0 \leq \varepsilon$, with the notation of Definition 1.1.*

Proof. Let $\varepsilon' > \varepsilon$ arbitrary, and assume by contradiction that ε' does not have the finite rotation property. Then there exists a non-continuous additive homomorphism $\gamma : \mathbb{R}^2 \rightarrow \mathbb{T}^n$ such that $|\gamma(\mathbf{u})| \geq \varepsilon'$ for all unit vectors \mathbf{u} , by Lemma 4.1. Let $g : \mathbb{C} \rightarrow \mathbb{T}^n$ be the function defined by $g(z) = (\gamma(\mathbf{u}_1 z), \dots, \gamma(\mathbf{u}_n z))$. Then $g(z) \in S \cap [\varepsilon', 1 - \varepsilon']^n \subset X_\varepsilon$ for all unit vectors z . However, if \mathbb{D} denotes the closed unit disk then $g(2\mathbb{D}) = g(\mathbb{C}_1) - g(\mathbb{C}_1) \subset X_\varepsilon - X_\varepsilon$ should be dense in \mathbb{T}^n by Lemma 4.2, a contradiction. \square

Improved upper bounds on ε_0 follow immediately.

Corollary 4.4. *With the notation of Definition 1.1 we have $\varepsilon_0 \leq \frac{1}{4}$.*

Proof. Let $\delta > 0$ be arbitrary and apply Theorem 4.3 with $\varepsilon = \frac{1}{4} + \delta$ and $\mathbf{u}_1 = 1$. Then $X_\varepsilon = (\varepsilon, 1 - \varepsilon)$, and $X_\varepsilon - X_\varepsilon$ is not dense in \mathbb{T} . Therefore, $\varepsilon_0 \leq \varepsilon$, and hence $\varepsilon_0 \leq \frac{1}{4}$. \square

A more elaborate argument gives the following improvement.

Corollary 4.5. *With the notation of Definition 1.1 we have $\varepsilon_0 \leq \frac{1}{5}$.*

Proof. Let $\delta > 0$ arbitrary and apply Theorem 4.3 with $\varepsilon = \frac{1}{5} + \delta$ and $\mathbf{u}_1 = 1, \mathbf{u}_2 = e^{2\pi i/3}, \mathbf{u}_3 = e^{4\pi i/3}$. In this case $S \subset \mathbb{T}^3$ is a two-dimensional torus, which can be identified with $[0, 1]^2$ via the projection mapping $(s_1, s_2, s_3) \mapsto (s_1, s_2)$. Elementary calculations give the exact position of the region X_ε : it is the union of two triangles T_1 and T_2 with coordinates of vertices $(\varepsilon, \varepsilon), (\varepsilon, 1 - 2\varepsilon), (1 - 2\varepsilon, \varepsilon)$ and $(1 - \varepsilon, 1 - \varepsilon), (1 - \varepsilon, 2\varepsilon), (2\varepsilon, 1 - \varepsilon)$, respectively. However, it is easy to check that these triangles are "too small" in the sense that $(T_1 \cup T_2) - (T_1 \cup T_2)$ is not dense in $[0, 1]^2$. Therefore, $\varepsilon_0 \leq \varepsilon$, and hence $\varepsilon_0 \leq \frac{1}{5}$. \square

We conclude this note with two remarks.

First, notice that the proofs of Corollaries 4.4 and 4.5 follow the same pattern. If $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\varepsilon > 0$ correspond to a covering of \mathbb{R}^2 , we slightly decrease the half-width of the strips to some $\rho < \varepsilon$, so that the non-covered region is still small enough, and then apply Theorem 4.3 to conclude that $\varepsilon_0 \leq \rho$. In Corollary 4.4 this was done for the trivial covering $\mathbf{u}_1 = 1, \varepsilon = \frac{1}{2}$ with the choice $\rho = \frac{1}{4}$. In Corollary 4.5 we used

the covering $\mathbf{u}_1 = 1, \mathbf{u}_2 = e^{2\pi i/3}, \mathbf{u}_3 = e^{4\pi i/3}$ for $\varepsilon = \frac{1}{3}$, with the choice $\rho = \frac{1}{5}$. It is plausible that this can be done for *every* covering, thus reducing the value of ε ever further. This would mean that the set of ε 's having the finite rotation property is open. However, it would still not prove that $\varepsilon_0 = 0$.

Second, Theorem 4.3 gives us the possibility to re-consider periodic coverings. For instance, let $\varepsilon < \frac{1}{2}$ and take all Pythagorean triples $m_j^2 + k_j^2 = n_j^2$ with $n_j \leq N$ for some fixed N . We know from Section 2 that the corresponding angles will not give a covering for ε . However, it is possible that the non-covered part X of the plane is small enough so that Theorem 4.3 can be invoked to conclude that ε has the finite rotation property. In fact, it is possible that this argument works for any $\varepsilon > 0$ if we choose N large enough, but we could not prove it so far.

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