TOWARDS A CLASSIFICATION OF 6×6 COMPLEX HADAMARD MATRICES

MÁTÉ MATOLCSI, FERENC SZÖLLŐSI

ABSTRACT. Complex Hadamard matrices have received considerable attention in the past few years due to their appearance in quantum information theory. While a complete characterization is currently available only up to order 5 (in [5]), several new constructions of higher order matrices have appeared recently [4, 12, 2, 7, 11]. In particular, the classification of *self-adjoint* complex Hadamard matrices of order 6 was completed by Beuachamp and Nicoara in [2], providing a previously unknown non-affine one-parameter orbit. In this paper we classify all *dephased, symmetric* complex Hadamard matrices *with real diagonal* of order 6. Furthermore, relaxing the condition on the diagonal entries we obtain a new non-affine one-parameter orbit connecting the Fourier matrix F_6 and Diţă's matrix D_6. This answers a recent question of Bengtsson & al. in [3].

1. INTRODUCTION

Throughout this paper we will use the notation of [12, 13] for well-known complex Hadamard matrices such as F_6, D_6, B_6 etc.

Original interest in *complex* Hadamard matrices arose in connection with orthogonal pairs of maximal Abelian ∗-subalgebras (MASA's) of the $n \times n$ matrices [5, 10, 6, 8, 9]. Subsequently, it was realized in [14] that complex Hadamard matrices also play an essential role in constructions of teleportation and dense coding schemes in quantum information theory. This fact has given a new boost to the study of complex Hadamard matrices in recent years. On the one hand, several new and general constructions of such matrices have appeared [4, 12, 7, 11]. On the other hand it is natural to try to *fully classify* complex Hadamard matrices of small order, as such characterization is currently available only up to order 5 in [5]. Recently some progress has been made in the 6×6 case in [2] where all *self-adjoint* complex Hadamard matrices are characterized, and in [3] where numerical evidence is given of the existence of a conjectured 4-parameter family. While an algebraic form of such a 4-parameter family (if it exists at all) remains out of reach, in this paper we present a *previously unknown non-affine one-parameter family* of 6×6 complex Hadamard matrices which *connects the Fourier matrix F_6 and Diţă's matrix D_6*. This result complements the recent catalogue [12] and answers a question of [3], proving that apart from the isolated matrix S_6 the set of known 6×6 Hadamard matrices is *connected*.

It is also important to mention that the 6×6 case is distinguished as the smallest dimension where the maximum number of mutually unbiased bases (MUBs) is not known. It is well-known that if d is a prime power than the maximal number of MUBs in $\mathbb{C}^d$ is $d + 1$. The existence of MUBs is equivalent to the existence of $d \times d$ complex Hadamard matrices

---

satisfying certain conditions (see e.g. [12]). For the current status of MUB-related problems and, in particular, the case $d = 6$ we refer to [3] and references therein. The recent discovery of the new family in [2], and the results of this paper may well be useful in the resolution of the MUB problem in dimension 6, and may give an indication to the maximal number of MUB's in dimension $d = pq$.

Throughout the paper we restrict attention to *dephased, symmetric* complex Hadamard matrices of order 6 (the standard terminology dephased meaning the normalization condition that all entries of the first row and column are $+1$). It is quite natural to study the symmetric case for two reasons. First, the results of [2] show that it is hopeful to obtain closed algebraic expressions if we require the matrix to satisfy certain symmetry assumptions (in [2] the self-adjoint case was classified). Second, the inspection of known 6×6 complex Hadamard matrices shows that many of them, such as F_6, D_6, C_6, S_6, are equivalent to a symmetric one (throughout the paper we use the standard notion of equivalence (see e.g. [12]), i.e. H_1 and H_2 are equivalent, $H_1 \cong H_2$, if $H_1 = D_1 P_1 H_2 P_2 D_2$ with unitary diagonal matrices D_1, D_2 and permutation matrices P_1, P_2). These two facts suggested that the set of symmetric Hadamard matrices of order 6 is on the one hand 'small' enough to be described in algebraic form and, on the other hand, 'rich' enough to contain interesting families of matrices. However, this intuitive approach turned out to be a little too optimistic in the first respect, and we needed to put further restrictions on the diagonal elements so that our algebraic calculations come to a comprehensible end. Accordingly, the outline of the paper is as follows. In Section 2 we fully classify *dephased, symmetric* complex Hadamard matrices of order 6 *with real diagonal*. It turns out that under this restriction well-known matrices emerge only. Therefore, in Section 3 we relax the condition on some diagonal entries and this leads to the discovery of a new non-affine one-parameter family.

2. Symmetric matrices with real diagonal

First we recall a simple but extremely useful result of [5] (see also [2, Lemma 2.6]).

Lemma 2.1. *Let u, v, s, t be complex numbers on the unit circle. Then*
$(u + v)(\overline{s} + \overline{t})(\overline{u}s + \overline{v}t) \in \mathbb{R}$. $\square$

We will also need the following elementary facts. In a dephased Hadamard matrix the sum of the entries in each row is 0 (except for the first row where the sum is, of course, n). Given a row $(x_1, x_2, x_3, x_4, x_5, x_6)$ we distinguish two possibilities. First, if $\Sigma := \frac{x_1+x_2+x_3+x_4}{2} = 0$ then $x_5 = -x_6$. Second, and more importantly, if $\Sigma \neq 0$ and $|\Sigma| \leq 1$ then the last two coordinates are determined (up to change of order) as

$$(1) \qquad\qquad x_{5,6} = -\Sigma \pm \mathbf{i} \frac{\Sigma}{|\Sigma|} \sqrt{1 - |\Sigma|^2}$$

The point is that $-2\Sigma = x_5 + x_6$ and it is easy to see geometrically that x_5 and x_6, being unit vectors, are determined as above.

The main result of this section is the following

Theorem 2.2. *Let H be a dephased, symmetric complex Hadamard matrix of order 6 with real diagonal. Then H is equivalent to S_6 or D_6.*

The proof is based on Lemma 2.1, and some considerations similar to those in [2].

The diagonal elements of H belong to $\{-1, 1\}$ by assumption. It is clear that there are either at least four 1's in the diagonal or at most three. Therefore, after a possible

permutation of the rows and columns it is enough to consider the following two possibilities for the diagonal of H:

$$(2) \qquad Diag(H) \in \{(1, 1, 1, 1, *, *), (1, -1, -1, -1, *, *)\},$$

where the $*$'s stand for ± 1.

Lemma 2.3. *Let H be a 6×6 symmetric complex Hadamard matrix of the form:*

$$(3) \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x & y & * & * \\ 1 & x & 1 & z & * & * \\ 1 & y & z & 1 & * & * \\ 1 & * & * & * & * & * \\ 1 & * & * & * & * & * \end{bmatrix}$$

Then

   (a) two of x, y, z must be equal.
   (b) H is equivalent to S_6.

(Note that we do not assume here that the last two diagonal entries are real; it is already implied by the above form.)

Proof. First we prove (a). Let us denote $h_{2,5} = u, h_{2,6} = v, h_{3,5} = s, h_{3,6} = t$. We will use Haagerup's idea as in Lemma 2.1. By the orthogonality relations of rows 1, 2, 3 we have

$$(4) \qquad 2 + x + y = -(u + v)$$

$$(5) \qquad 2 + \bar{x} + \bar{z} = -(\bar{s} + \bar{t})$$

$$(6) \qquad 1 + x + \bar{x} + z\bar{y} = -(s\bar{u} + t\bar{v}).$$

Now, Lemma 2.1 implies

$$(7) \qquad (2 + x + y)(2 + \bar{x} + \bar{z})(1 + x + \bar{x} + z\bar{y}) \in \mathbb{R}.$$

By similar arguments we obtain

$$(8) \qquad (2 + \bar{x} + \bar{y})(2 + y + z)(1 + \bar{y} + y + \bar{z}x) \in \mathbb{R}$$

$$(9) \qquad (2 + x + z)(2 + \bar{y} + \bar{z})(1 + z + \bar{z} + y\bar{x}) \in \mathbb{R}.$$

After summing up these three expressions and eliminating real terms we get

$$(10) \qquad \bar{x}^2 y + \bar{x} y^2 + x\bar{z}^2 + x^2\bar{z} + \bar{y}^2 z + \bar{y} z^2 + 8(\bar{x}y + x\bar{z} + \bar{y}z) \in \mathbb{R}$$

Since a complex number is real if and only if it is equal to its conjugate, we obtain an equality if we replace each variable by its conjugate (i.e. its reciprocal) in the above expression. The resulting equality can then be rearranged by simple algebra to yield

$$(11) \qquad (x - y)(x - z)(y - z)(xy + yz + zx + 8xyz + x^2 yz + xy^2 z + xyz^2) = 0$$

The last factor in the product is clearly non-zero by the triangle inequality (one term has modulus 8, and the others have modulus 1). This proves (a).

Now we turn to (b). It is easy to see that all $x = y$, $x = z$, $y = z$ lead to equivalent Hadamard matrices by permutation, thus we can assume without loss of generality that $x = y$.

Now substitute back to (7) and eliminate all real terms to get

$$(12) \qquad\qquad x^2 + \overline{xz} + \overline{x}z \in \mathbb{R}.$$

Therefore this expression equals its conjugate and we obtain

$$(13) \qquad\qquad x^2 + \overline{xz} + \overline{x}z - \overline{x}^2 - xz - x\overline{z} = 0$$

which yields

$$(14) \qquad\qquad \left(x^2 - 1\right)\left(z + x^2 z - x - xz^2\right) = 0$$

Here $x = y = 1$ is clearly a contradiction, because the first two rows of H cannot be orthogonal.

To show that $x = y = -1$ is also impossible we need to consider two subcases. If $x = y = -1 = z$ then the rows of the leading 4×4 minor of H are already mutually orthogonal, therefore the last two entries of the first four rows of H should also be mutually orthogonal, which is clearly impossible. If $x = y = -1 \neq z$ then H must be equivalent to a matrix of the following form

$$(15) \qquad\qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & u & -u \\ 1 & -1 & 1 & z & -z & -1 \\ 1 & -1 & z & 1 & -1 & -z \\ 1 & u & -z & -1 & * & * \\ 1 & -u & -1 & -z & * & * \end{bmatrix}.$$

The last two entries of rows 3, 4 are determined by (1), and the order of -1 and z in the fourth row is forced by the orthogonality of rows 3-4. The same orthogonality now implies $z = \omega$ or $z = \omega^2$ (with ω being the third root of unity). If $z = \omega$ then the orthogonality of rows 2 and 3 implies that $u = -\frac{\mathbf{i}}{\sqrt{3}}$ which is a contradiction. $z = \omega^2$ implies $u = \frac{\mathbf{i}}{\sqrt{3}}$, again a contradiction.

Therefore in (14) we must have $(z + x^2 z - x - xz^2) = (z - x)(1 - xz) = 0$, which implies $z = x$ or $z = \overline{x}$. The case $z = x = y$ leads again to contradiction due to the following reasons. First, we cannot have $x = y = z = -1$ as argued already above. Second, if $x = y = z \neq -1$ then (1) implies that two of the rows 2, 3, 4 must contain the same two entries in the last two places. However, those two rows then contain the same element in four places, therefore they cannot be orthogonal.

Thus, we must have $y = x \neq \pm 1$, $z = \overline{x}$. In this case H must be equivalent with the following form

(16)
$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x & x & * & * \\ 1 & x & 1 & \overline{x} & u & v \\ 1 & x & \overline{x} & 1 & v & u \\ 1 & * & u & v & * & * \\ 1 & * & v & u & * & * \end{bmatrix}$$

where $\Re[x] \leq 0$ due to the orthogonality of rows 1 and 3. After conjugating H if necessary (and noting that S_6 is equivalent to its conjugate) we can also assume that $\Im[x] \geq 0$. The third and fourth rows are being forced by (1) as follows (we are free to choose the order due to permutation equivalence):

(17)
$$u = -1 - \Re[x] + \mathbf{i}\sqrt{-\Re[x]^2 - 2\Re[x]},$$

(18)
$$v = -1 - \Re[x] - \mathbf{i}\sqrt{-\Re[x]^2 - 2\Re[x]}.$$

Now, due to the orthogonality of rows 3, 4 we get $4\Re[x]^2 + 10\Re[x] + 4 = 0$ which gives the only possible solution $\Re[x] = -\frac{1}{2}$ and, by $\Im[x] \geq 0$, we get $x = \omega$. Then by (17) and (18) $u = \omega$, $v = \omega^2$, and

(19)
$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & * & * \\ 1 & \omega & 1 & \omega^2 & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ 1 & * & \omega & \omega^2 & * & * \\ 1 & * & \omega^2 & \omega & * & * \end{bmatrix}$$

All the remaining entries are determined uniquely (using (1) and orthogonality) and we obtain

(20)
$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ 1 & \omega^2 & \omega & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega^2 & \omega & \omega & 1 \end{bmatrix}$$

This matrix is clearly equivalent to Tao's matrix S_6 (we note here, that this matrix was published earlier in [1], page 104). $\qquad\square$

Having classified dephased, symmetric Hadamard matrices with diagonal $(1, 1, 1, 1, *, *)$ we can assume that the number of 1's in the diagonal are at most three, in other words, the diagonal is $(1, -1, -1, -1, *, *)$.

Lemma 2.4. *Let H be a 6×6 symmetric complex Hadamard matrix of the form*

(21)
$$
H = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & x & y & * & * \\
1 & x & -1 & z & * & * \\
1 & y & z & -1 & * & * \\
1 & * & * & * & * & * \\
1 & * & * & * & * & *
\end{bmatrix}
$$

Then H is equivalent to D_6. (Note that we do not assume here that the last two diagonal entries are real.)

Proof. We can assume that $\Im[x] \geq 0$ (by conjugating all entries of H if necessary, and noting that D_6 is equivalent to its conjugate).

First assume that two of x, y, z are equal, say $x = y$ (we are free to choose due to permutation equivalence) and, furthermore, $z = -x$. Then H is equivalent to

(22)
$$
H = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & x & x & -x & -x \\
1 & x & -1 & -x & u & -u \\
1 & x & -x & -1 & v & -v \\
1 & -x & u & v & * & * \\
1 & -x & -u & -v & * & *
\end{bmatrix}
$$

By the orthogonality of rows 3, 4

(23)
$$
1 + 1 + \overline{x} + x + u\overline{v} + u\overline{v} = 0,
$$

which yields $u\overline{v} \in \mathbb{R}$, therefore $v = u$ or $v = -u$. The former is not possible due to (23), therefore $v = -u$. Then, using again (23) we get $x + \overline{x} = 0$ and hence, by the assumption $\Im[x] \geq 0$, $x = \mathbf{i}$.

Then, using (1) we obtain that the last two entries of rows 5, 6 must be -1 and $\mathbf{i}$ in some order. The same order in both rows is not possible because then these rows would agree in four entries and could not be orthogonal. Therefore we have two possibilities:

(24) $H = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} \\
1 & \mathbf{i} & -1 & -\mathbf{i} & u & -u \\
1 & \mathbf{i} & -\mathbf{i} & -1 & -u & u \\
1 & -\mathbf{i} & u & -u & -1 & \mathbf{i} \\
1 & -\mathbf{i} & -u & u & \mathbf{i} & -1
\end{bmatrix}$, or $H = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} \\
1 & \mathbf{i} & -1 & -\mathbf{i} & u & -u \\
1 & \mathbf{i} & -\mathbf{i} & -1 & -u & u \\
1 & -\mathbf{i} & u & -u & \mathbf{i} & -1 \\
1 & -\mathbf{i} & -u & u & -1 & \mathbf{i}
\end{bmatrix}$.

In the first case, by the orthogonality of rows 4, 5 we get $u = \pm\mathbf{i}$. Both choices lead to Hadamard matrices equivalent to D_6.

In the second case the orthogonality of rows 4, 5 yields $u = \pm 1$. Both choices lead to Hadamard matrices equivalent to D_6.

Let us now turn to the case when $x = y$, but $z \neq -x$. We will show that it is not possible. The last two entries of row 3 must be $-x$ and $-z$ and we are free to choose the order due to

permutation equivalence. Therefore,

$$(25) \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & x & -x & -x \\ 1 & x & -1 & z & -x & -z \\ 1 & x & z & -1 & * & * \\ 1 & -x & -x & * & * & * \\ 1 & -x & -z & * & * & * \end{bmatrix}$$

By the orthogonality of rows 2, 3 it we get

$$(26) \qquad 1 - \overline{x} - x + x\overline{z} + 1 + x\overline{z} = 0,$$

thus $x\overline{z} \in \mathbb{R}$ implying $x = z$ or $x = -z$. The first case is not possible by (26), while the second contradicts our current assumptions.

Now we turn to the case when x, y, z are all distinct. Again, we will show that it is not possible. We need to distinguish two subcases. First we assume that one variable is the negative of another, say $y = -x$ (we are free to choose due to permutation equivalence). By assumption we cannot have $z = x$ or $z = -x$ (in the latter case $z = y$ would hold). Therefore we have two choices to fill up the last two entries of rows 3, 4:

$$(27) \quad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & -x & u & -u \\ 1 & x & -1 & z & -x & -z \\ 1 & -x & z & -1 & x & -z \\ 1 & u & -x & x & * & * \\ 1 & -u & -z & -z & * & * \end{bmatrix} \quad \text{or} \quad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & -x & u & -u \\ 1 & x & -1 & z & -x & -z \\ 1 & -x & z & -1 & -z & x \\ 1 & u & -x & x & * & * \\ 1 & -u & -z & -z & * & * \end{bmatrix}$$

In the first case the orthogonality of rows 3, 4 implies $z = \pm\mathbf{i}$ and we can assume (by conjugation if necessary) that $z = \mathbf{i}$. Then the orthogonality of rows 2, 3 and 2, 4 yield the equalities $1 - x - \overline{x} - \overline{x}\mathbf{i} - \overline{u}x + \overline{u}\mathbf{i} = 0$ and $1 + x + \overline{x}\mathbf{i} + \overline{x} + \overline{u}x + \overline{u}\mathbf{i} = 0$ which, after summation, imply $u = -\mathbf{i}$. Substituting back $u = -\mathbf{i}$ we get $x = \pm\mathbf{i}$. But this implies $z = x$ or $z = y(= -x)$ which contradicts our current assumptions.

In the second case of (27) the orthogonality of rows 3, 4 implies $1 - 1 - \overline{z} - z + x\overline{z} - \overline{x}z = 0$, which yields $x\overline{z} - \overline{x}z \in \mathbb{R}$. This is only possible if $x\overline{z} \in \mathbb{R}$, that is $x = z$ or $-x = z = y$, which are both excluded by assumption.

Lastly, assume that all x, y, z are all distinct and none of them is the negative of another. By formula (1) there are the following four different possibilities to fill out the last two entries of rows 2, 3, 4 (we are free to fix the order of $-x$ and $-y$ in row 2 due to permutation equivalence):

$$(28) \qquad \begin{bmatrix} 1 & 1 \\ -x & -y \\ -x & -z \\ -y & -z \\ * & * \\ * & * \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -x & -y \\ -x & -z \\ -z & -y \\ * & * \\ * & * \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -x & -y \\ -z & -x \\ -y & -z \\ * & * \\ * & * \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -x & -y \\ -z & -x \\ -z & -y \\ * & * \\ * & * \end{bmatrix}$$

Now we analyze these cases separately.

CASE[1]: consider the first possibility listed in (28), that is

$$(29) \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & y & -x & -y \\ 1 & x & -1 & z & -x & -z \\ 1 & y & z & -1 & -y & -z \\ 1 & -x & -x & -y & * & * \\ 1 & -y & -z & -z & * & * \end{bmatrix}$$

Taking the inner product of rows 2, 3

$$(30) \qquad 1 - \overline{x} - x + y\overline{z} + 1 + y\overline{z} = 0$$

Thus, $y\overline{z} \in \mathbb{R}$ and hence $z = y$ or $z = -y$ which are both excluded by assumption.

CASE[2]: the second possibility listed in (28). Consider rows 2, 4 and apply the same simple argument as in CASE[1] above.

CASE[3]: the third possibility listed in (28). The orthogonality of rows 2, 3 implies

$$(31) \qquad 1 - \overline{x} - x + y\overline{z} + x\overline{z} + \overline{x}y = 0,$$

which means that $\overline{x}y + x\overline{z} + y\overline{z} \in \mathbb{R}$. Therefore this expression equals its conjugate, i.e.

$$(32) \qquad \overline{x}y + x\overline{z} + y\overline{z} = x\overline{y} + \overline{x}z + \overline{y}z.$$

Using that conjugates are the same as reciprocals this equation is equivalent to

$$(33) \qquad (x + y)(x + z)(y - z) = 0,$$

which contradicts our assumptions.

CASE[4]: the fourth possibility listed in (28). Consider rows 3, 4 and apply the same simple argument as in CASE[1] above. $\qquad \square$

3. A NEW FAMILY OF 6×6 COMPLEX HADAMARD MATRICES

In Theorem 2.2 we have classified all dephased symmetric Hadamard matrices with real diagonal. Furthermore, from Lemmas 2.3 and 2.4 we see that it was, in fact, enough to specify four real entries in the diagonal. It is then natural to investigate the two remaining real options for the first four entries of the diagonal, i.e. the cases $DiagH \in (1, -1, 1, 1, *, *)$ and $DiagH \in (1, -1, -1, 1, *, *)$.

Along the lines of Lemmas 2.3 and 2.4 a case-by-case argument shows that *there exists no dephased symmetric complex Hadamard matrix with diagonal* $(1, -1, -1, 1, *, *)$. We do not include the details of this fruitless calculation.

The last remaining case, $DiagH \in (1, -1, 1, 1, *, *)$, turns out to be the most interesting one. Unfortunately we do not have a full classification in this case, but we are able to obtain new matrices nevertheless. For some preliminary calculations we disregard the last $+1$ entry in the diagonal, and assume only that $DiagH \in (1, -1, 1, *, *, *)$. Due to the presence of the -1 in the second row, the remaining entries must be $x, -x$ and $y, -y$, and H takes the form (up to permutation equivalence)

$$(34) \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & -y & y & -x \\ 1 & x & 1 & a & b & c \\ 1 & -y & a & * & * & * \\ 1 & y & b & * & * & * \\ 1 & -x & c & * & * & * \end{bmatrix}.$$

Using the orthogonality of rows 1, 2, 3 we apply Lemma 2.1 as follows.

$$(35) \qquad x + y = -\left(-y - x \right),$$

$$(36) \qquad 2 + \overline{x} + \overline{b} = -\left(\overline{a} + \overline{c} \right),$$

$$(37) \qquad 1 - x + \overline{x} + b\overline{y} = -\left(-a\overline{y} - c\overline{x} \right),$$

therefore

$$(38) \qquad (x + y)\left(2 + \overline{x} + \overline{b} \right)\left(1 - x + \overline{x} + b\overline{y} \right) \in \mathbb{R}$$

After expanding and eliminating the real entries one gets:

$$(39) \qquad -2x^2 - 2xy + 2\overline{x}y + \overline{x}^2 y - x^2 \overline{b} - xy\overline{b} + x\overline{y}b + y + b \in \mathbb{R}$$

This expression therefore equals its conjugate and by simple algebra we get

$$(40) \qquad (y + x)\left[b^2(1 + x^2) + b(2 - x^3 - 2x^3 y - 2x^2 - x + y + x^2 y + 2xy) - xy(1 + x^2) \right] = 0$$

Unfortunately, we do not know how to handle the case when the second factor equals zero, therefore we need to settle for the *simplifying assumption* $y = -x$. We remark here, however, that there exist non-trivial solutions with $y \neq -x$ too, such as the permuted version of Björck's cyclic matrix

$$(41) \qquad C_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & d^2 & -d & d & -d^2 \\ 1 & d^2 & 1 & -d^3 & -\overline{d} & d^2 \\ 1 & -d & -d^3 & -d^3 & -d & d^4 \\ 1 & d & -\overline{d} & -d & \overline{d} & -1 \\ 1 & -d^2 & d^2 & d^4 & -1 & -d^4 \end{bmatrix},$$

where $d = \frac{1 - \sqrt{3}}{2} + \mathbf{i} \cdot \sqrt{\frac{\sqrt{3}}{2}}$.

Having made the assumption $y = -x$ the matrix H will now be determined up to permutation equivalence and possible conjugation. Now, H takes the form

$$(42) \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & x & -x & -x \\ 1 & x & 1 & a & b & c \\ 1 & x & a & t & u & v \\ 1 & -x & b & u & p & q \\ 1 & -x & c & v & q & r \end{bmatrix}$$

From the orthogonality of rows 2, 3 and 1, 3 we get:

$$(43) \qquad 1 - \overline{x} + x + x\overline{a} - x\overline{b} - x\overline{c} = 0$$

$$(44) \qquad 1 + \overline{x} + 1 + \overline{a} + \overline{b} + \overline{c} = 0$$

Multiplying (44) by x and then summing up and using $\overline{x} = 1/x$ we get

$$(45) \qquad a = \frac{x^2 - 2x - 3}{2}.$$

This equation does have two solutions such that both x and a are on the unit ball,

$$(46) \qquad x_{1,2} = \frac{1 - \sqrt{13}}{3} \pm \mathbf{i}\frac{\sqrt{-5 + 2\sqrt{13}}}{3}$$

Let us take $x = x_1$ (the other choice leads to the conjugate matrix), and hence

$$(47) \qquad a = -\frac{7 - \sqrt{13}}{9} - \mathbf{i}\frac{\sqrt{19 + 14\sqrt{13}}}{9}$$

Now, since $2 + x + a \neq 0$ we can apply (1) and obtain (up to change of order, which we are free to choose due to permutation equivalence)

$$(48) \qquad b = \frac{-14 + 2\sqrt{13} - \sqrt{-58 + 34\sqrt{13}}}{18} - \mathbf{i}\frac{\sqrt{134 + 22\sqrt{13} - 8\sqrt{-2446 + 730\sqrt{13}}}}{18}$$

$$(49) \qquad c = \frac{-14 + 2\sqrt{13} + \sqrt{-58 + 34\sqrt{13}}}{18} + \mathbf{i}\frac{\sqrt{134 + 22\sqrt{13} + 8\sqrt{-2446 + 730\sqrt{13}}}}{18}$$

Next we find t, u and v. The orthogonality of rows 1, 4 and 2, 4 yield

$$(50) \qquad 1 + x + a + t + u + v = 0,$$

$$(51) \qquad 1 - x + a\overline{x} + t\overline{x} - u\overline{x} - v\overline{x} = 0.$$

Multiplying (51) by x and then summing up we obtain

$$(52) \qquad t = \frac{x^2 - 2x - 1 - 2a}{2}.$$

Substituting the values of x and a we get $t = 1$.

Then, using (1) we obtain (the order being determined by the orthogonality of rows 3, 4)

$$(53) \qquad u = c$$

$$(54) \qquad v = b$$

Finally we can use (1) once again to complete rows 5 and 6 as (the order being determined by orthogonality of rows 4, 5, 6):

$$(55) \qquad p = r = 3 - \sqrt{13} - \mathbf{i}\sqrt{-21 + 6\sqrt{13}}$$

$$(56) \qquad q = \frac{-19 + 4\sqrt{13}}{9} + \mathbf{i}\frac{2\sqrt{-122 + 38\sqrt{13}}}{9}$$

Therefore, we have obtained

$$(57) \qquad M_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & x & -x & -x \\ 1 & x & 1 & a & b & c \\ 1 & x & a & 1 & c & b \\ 1 & -x & b & c & p & q \\ 1 & -x & c & b & q & p \end{bmatrix},$$

where x, a, b, c, p, q are determined by (46), (47), (48), (49), (55) and (56), respectively. It is easy to check that M_6 is indeed Hadamard. What we have shown above is that *up to permutation equivalence M_6 and its conjugate M_6^* are the only dephased symmetric Hadamard matrices with diagonal $(1, -1, 1, *, *, *)$ and second row consisting of the elements* $(1, 1, x, x, -x, -x)$.

We will now proceed to show that M_6 is not contained in any of the previously known 6×6 families. We need the following trivial

Lemma 3.1. *If a symmetric complex Hadamard matrix H is equivalent to a self-adjoint one, then it is also equivalent to its own conjugate i.e. $H \cong \overline{H}$.*

Proof. Let H be a symmetric complex Hadamard matrix and suppose that it is equivalent to a self-adjoint one, say to $A = A^*$. Then there are unitary diagonal D_1, D_2 and permutational matrices P_1, P_2 such that

$$(58) \qquad P_1 D_1 H D_2 P_2 = A = A^* = P_2^* D_2^* H^* D_1^* P_1^*$$

By multiplying both sides with $D_2 P_2$ from the left and $P_1 D_1$ from the right we get:

$$(59) \qquad D_2 P_2 P_1 D_1 H D_2 P_2 P_1 D_1 = H^* = \overline{H}$$

This clearly says that $H \cong \overline{H}$. $\qquad \square$

As a consequence we have

Proposition 3.2. *M_6 and M_6^* are not equivalent to any previously known complex Hadamard matrix of order 6.*

Proof. We will use the Haagerup Λ-set of a matrix $H = [h_{jk}]$, defined as

$$(60) \qquad \Lambda_H = \{ h_{ij} \overline{h}_{kj} h_{kl} \overline{h}_{il} \ \text{ for all } \ 1 \le i, j, k, l \le 6 \}.$$

It is well-known that Λ_H is invariant under equivalence (see [5]).

The list of known 6×6 Hadamard matrices is as follows. The Fourier family $F_6^{(2)}(a, b)$ and its transposed $\left(F_6^{(2)}(a, b) \right)^T$, the Diţă family $D_6^{(1)}(c)$, and Tao's matrix S_6 are listed in [12]. The recently discovered non-affine family $B_6^{(1)}(a)$ is given in [2].

M_6 is clearly inequivalent to S_6 due to the Haagerup Λ-set being different. M_6 is inequivalent to any matrix in $F_6^{(2)}(a, b)$ since the third root of unity $\omega \in \Lambda_{F_6^{(2)}(a,b)}$ for every matrix in that family (i.e. for every a, b), while $\omega \notin \Lambda_{M_6}$. The same is true for the transposed family $\left(F_6^{(2)}(a, b) \right)^T$. M_6 is inequivalent to any matrix in $D_6^{(1)}(c)$ since $\mathbf{i} \in \Lambda_{D_6^{(1)}(c)}$ for every matrix in that family, while $\mathbf{i} \notin \Lambda_{M_6}$. Finally, using Lemma 3.1, M_6 is inequivalent to any of the matrices contained in the self-adjoint non-affine family $B_6^{(1)}(a)$ since M_6 is inequivalent to

$\overline{M}_6$, which can be seen as follows. Let a be as in M_6. One can easily check that $\overline{a}^2 \in \Lambda_{M_6}$ and $a^2 \notin \Lambda_{M_6}$, while $\overline{a}^2 \notin \Lambda_{\overline{M}_6}$ and $a^2 \in \Lambda_{\overline{M}_6}$.

The same proof works for M_6^*, too.   □

We will now generalize our discrete result above and construct a continuous one-parameter family stemming from M_6. As M_6 is in block-matrix form it is quite natural to try and replace the 1's on the main diagonal by some parameter d and consider dephased symmetric matrices of the following form:

$$(61) \qquad M_6(x) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & x & x & -x & -x \\ 1 & x & d & a & b & c \\ 1 & x & a & d & c & b \\ 1 & -x & b & c & p & q \\ 1 & -x & c & b & q & p \end{bmatrix},$$

The orthogonality of rows 2, 3 and 1, 3 imply

$$(62) \qquad 1 - x + d\overline{x} + a\overline{x} - b\overline{x} - c\overline{x} = 0$$

$$(63) \qquad 1 + x + d + a + b + c = 0$$

Multiplying (62) by x and then summing up we get

$$(64) \qquad a = \frac{x^2 - 2x - 1}{2} - d.$$

It is easy to see that $0 < |x^2 - 2x - 1| < 4$ for each choice of x on the unit circle, therefore we can use (64) to apply (1) to obtain the values of a and d as follows (we are free to choose the order due to permutation equivalence):

$$(65) \qquad a = \frac{x^2 - 2x - 1}{4} - \mathbf{i} \frac{(x^2 - 2x - 1)\sqrt{16 - |x^2 - 2x - 1|^2}}{4|x^2 - 2x - 1|}$$

$$(66) \qquad d = \frac{x^2 - 2x - 1}{4} + \mathbf{i} \frac{(x^2 - 2x - 1)\sqrt{16 - |x^2 - 2x - 1|^2}}{4|x^2 - 2x - 1|}$$

From (64) we see that $1 + x + a + d = \frac{x^2+1}{2}$ which vanishes if and only if $x = \pm\mathbf{i}$. We exclude $x = \pm\mathbf{i}$ from these considerations and remark that this case can be handled separately as in Lemma 2.4, and leads to $H \cong D_6$. For $x \neq \pm\mathbf{i}$ the values of b and c are determined uniquely by (1) as follows (we are free to choose the order due to permutation equivalence):

$$(67) \qquad b = -\frac{1 + x^2}{4} - \mathbf{i} \frac{(1 + x^2)\sqrt{16 - |1 + x^2|^2}}{4|1 + x^2|}$$

$$(68) \qquad c = -\frac{1 + x^2}{4} + \mathbf{i} \frac{(1 + x^2)\sqrt{16 - |1 + x^2|^2}}{4|1 + x^2|}$$

It is easy to check (rather by computer) that with these parametric choices the first four rows of H are mutually orthogonal to each other.

We evaluate $1 - x + b + c$ in order to use (1) again to determine p and q. The orthogonality of rows 1, 5 and 2, 5 imply

$$（69) \qquad 1 - x + b + c + p + q = 0$$

$$(70) \qquad 1 + x + \overline{x}b + \overline{x}c - \overline{x}p - \overline{x}q = 0.$$

Multiplying (70) by x and summing up we get

$$(71) \qquad 1 - x + b + c = \frac{-x^2 - 2x + 1}{2}$$

and we see that it does not vanish for unit vectors x. Therefore we can apply (1) to determine p and q as follows (the order now being forced by orthogonality):

$$(72) \qquad p = \frac{x^2 + 2x - 1}{4} + \mathbf{i}\frac{(x^2 + 2x - 1)\sqrt{16 - |x^2 + 2x - 1|^2}}{4|x^2 + 2x - 1|}$$

$$(73) \qquad q = \frac{x^2 + 2x - 1}{4} - \mathbf{i}\frac{(x^2 + 2x - 1)\sqrt{16 - |x^2 + 2x - 1|^2}}{4|x^2 + 2x - 1|}$$

It is easy to check by computer that the emerging matrix H is Hadamard for all $x \neq \pm\mathbf{i}$. Therefore we have obtained the following

Theorem 3.3. *There is a one parameter* symmetric *non-affine family of Hadamard matrices given by* (61), *with* $x = e^{it}$, $x \neq \pm\mathbf{i}$, *and* a, b, c, d, p, q *being given as in* (65), (67), (68), (66), (72), (73), *respectively.*

Finally we make the following observation which answers a question raised in [3] and shows that the set of currently known Hadamard matrices of order 6 is connected except for the isolated matrix S_6. In particular, the family $M_6(x)$ above connects the Fourier matrix F_6 and Diţă's matrix D_6.

Observation 3.4. $M_6(1) \cong F_6$, $\lim\limits_{t \to \frac{3\pi}{2}} M_6\left(e^{it}\right) \cong D_6$, *and finally,* $M_6\left(e^{\mathbf{i}\arccos\left(\frac{1-\sqrt{13}}{3}\right)}\right) = M_6$.

$$(74) \qquad \lim_{t \to \frac{3\pi}{2}} M_6\left(e^{it}\right) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & -1 & \mathbf{i} & -1 & 1 \\ 1 & -\mathbf{i} & \mathbf{i} & -1 & 1 & -1 \\ 1 & \mathbf{i} & -1 & 1 & -\mathbf{i} & -1 \\ 1 & \mathbf{i} & 1 & -1 & -1 & -\mathbf{i} \end{bmatrix},$$

$$(75) \qquad M_6(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & \omega^2 & \omega & \omega^2 & \omega \\ 1 & 1 & \omega & \omega^2 & \omega & \omega^2 \\ 1 & -1 & \omega^2 & \omega & -\omega^2 & -\omega \\ 1 & -1 & \omega & \omega^2 & -\omega & -\omega^2 \end{bmatrix}.$$

In summary, we have given a full classification of dephased symmetric complex Hadamard matrices of order 6 with real diagonal, showing that in this case the well-known matrices S_6 and D_6 emerge only. Furthermore, relaxing the reality condition on the diagonal entries we have been able to obtain a new non-affine family of dephased symmetric Hadamard matrices of order 6 which connects F_6 and D_6.

It would be interesting to see whether this family can be extended by further parameters. For example, it is natural to try to replace the second entry of the diagonal (-1) by a parameter h. However, currently we are unable to classify that case. It also remains to be seen whether this new family helps to increase the number of bases appearing in MUBs in $\mathbb{C}^6$.

REFERENCES

[1] S. S. Agaian: *Hadamard Matrices and Their Applications*, Lecture Notes in Mathematics, 1168, Springer-Verlag, Berlin, (1985).

[2] K. Beauchamp, R. Nicoara: Orthogonal maximal Abelian *-subalgebras of the 6×6 matrices, *preprint,* http://arxiv.org/abs/math/0609076

[3] I. Bengtsson, W. Bruzda, A. Ericsson, J.-A. Larsson, W. Tadej, K. Życzkowski: MUBs and Hadamards of Order Six, *preprint,* http://arxiv.org/abs/quant-ph/0610161

[4] P. Diţă, Some results on the parametrization of complex Hadamard matrices, *J. Phys. A*, **37** no. 20, 5355–5374. (2004)

[5] U. Haagerup, Ortogonal maximal Abelian *-subalgebras of $n \times n$ matrices and cyclic n-roots, Operator Algebras and Quantum Field Theory (Rome), Cambridge, MA International Press, (1996), 296–322.

[6] P. de la Harpe, V. F. R. Jones: Paires de sous-algebres semi-simples et graphes fortement reguliers, *C.A. Acad. Sci. Paris,* **311**, serie I (1990), 147-150.

[7] M. Matolcsi, J. Réffy, F. Szöllősi: Constructions of complex Hadamard matrices via tiling Abelian groups, *Open Syst. Inf. Dyn.,* to appear; (available at http://arxiv.org/abs/quant-ph/0607073)

[8] A. Munemasa, Y. Watatani: Orthogonal pairs of *-subalgebras and association schemes, *C.R. Acad. Sci. Paris,* **314**, serie I (1992), 329-331.

[9] R. Nicoara: A finiteness result for commuting squares of matrix algebras, *J. Operator Theory*, **55**, (2006), no. 2, 295–310.

[10] S. Popa: Orthogonal pairs of *-subalgebras in finite von Neumann algebras, *J. Operator Theory*, **9**, 253-268 (1983).

[11] F. Szöllősi: Parametrizing Complex Hadamard Matrices, *preprint,* http://arxiv.org/abs/math/0610297

[12] W. Tadej, K. Życzkowski. A concise guide to complex Hadamard matrices *Open Syst. Inf. Dyn.* **13**, 133-177. (2006)

[13] Website for complex Hadamard matrices: http://chaos.if.uj.edu.pl/˜karol/hadamard/

[14] R. F. Werner: All teleportation and dense coding schemes, *J. Phys. A*, **34**, (2001), 7081–7094.

MÁTÉ MATOLCSI: ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, POB 127, H-1364 BUDAPEST, HUNGARY.
  E-mail address: `matomate@renyi.hu`

FERENC SZÖLLŐSI: BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS (BUTE), H-1111, EGRY J. U. 1, BUDAPEST, HUNGARY.
  E-mail address: `szoferi@math.bme.hu`