ESTIMATES FOR THE DIMENSIONS OF NONNEGATIVE REALIZATIONS

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Abstract. The problem of nonnegative realizations of a transfer function (i.e. of a rational matrix function vanishing at infinity) is an important question from the engineering and a highly nontrivial one from the mathematical point of view. We give an upper estimate for the dimension of a nonnegative realization of a primitive scalar transfer function with simple pole at the spectral radius. We show how information on nonnegative realizations of the entries can be used for the upper estimation of the dimension of such realizations of a matrix transfer function.

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1. Introduction

Consider a rational matrix function H vanishing at infinity, i.e. let

$$H(z) := \frac{P_1 z^{n-1} + \cdots + P_n}{z^n - q_1 z^{n-1} - \cdots - q_n},$$

where the entries of the $l \times k$ matrices P_j and the numbers q_j ($j = 1, \ldots, n$) are real. In the theory of linear (time-invariant multiple input, multiple output [MIMO]) systems such a function is a *transfer function*, and a *realization of H of order (or, equivalently, of dimension) $r \in \mathbb{N}$* is a triple (C, A, B) of matrices of sizes $l \times r, r \times r, r \times k$ satisfying $H(z) = C(z - A)^{-1}B$ for every $z \in \mathbb{C}$ which is no eigenvalue of the matrix A. For the general theory of realizations we refer the reader to [GLR] or [Sont], and shall be concerned here with some problems of *nonnegative realizations*, where the matrices C, A, B are *entrywise* nonnegative. Note that they are also called (equivalently) *positive* realizations, and an elegant necessary and sufficient condition for their existence was proved in [Far].

For further basic questions of such realizations we refer to [FN1] and [FN2] and the references there, and to the recent excellent survey paper

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[BF] by Benvenuti and Farina. For some of the engineering applications see, e.g., the monograph by Farina and Rinaldi [FaRi].

Assuming that a nonnegative realization of H exists, it is well known (see e.g. [NM]) that the minimal order of such realizations can be much larger than the minimal order of *all* realizations (the *McMillan degree*). A nonnegative realization of H is called an *MP realization* if its order is minimal in the class of all nonnegative realizations of H. In general, the determination of the order of MP realizations of H seems to be hopeless. Therefore it is an interesting theoretical and practical problem to give bounds on this order (in this direction some results have been obtained for certain classes of transfer functions in [NM], [Ha]).

The remarkable paper [ADFB] proved the existence of a nonnegative realization for the case of a *primitive* scalar-valued transfer function H. The authors expressed on several occasions (see e.g. [BFA]) that it would be desirable, but they have not given any estimate (in the case of arbitrary poles in the interior of the "spectral disc") on the order of such a realization. In Section 2 of this paper we provide such an estimate in Theorem 2.2 for any *primitive* H with *simple pole at the spectral radius of* A in the *scalar* (i.e. $l = 1, k = 1$ or, equivalently, single input, single output [SISO]) case, and refer the reader to [ADFB] and [Far] for the connection of this very important case to the case of a general scalar function H.

It is well known that the existence of a nonnegative realization of a transfer function H is equivalent to the existence of a so called *system invariant polyhedral cone* (cf., e.g., [FN1]). It is also known (see [FN1, Prop. 2.2]) that the order of an MP realization is not greater than the minimal number of the edges ($\equiv$ generating rays) of system invariant polyhedral cones, and the former may be strictly less. In Theorem 3.1 of this paper we provide an estimate of the number of the edges of a system invariant polyhedral cone of a MIMO system based on the numbers of the edges of the corresponding cones for the entries (SISO systems) of the transfer function H. The estimates here are obtained with the help of some classical (but perhaps not widely known) inequalities of Minkowski and Weyl.

Now we shall fix some *terminology and notation.* A transfer function H vanishing at infinity is *primitive* (cf. [Far, Definition 2.1]) if it has a unique maximum modulus pole (the dominant pole) of some finite order at a positive number. If (C, A, B) is any realization of H, then the sequence of matrices

$$\{h_k := CA^kB;\ k = 0, 1, 2, \dots\}$$

is the *Markov sequence* or *impulse response* of H, and is clearly the sequence of the coefficients of the Laurent expansion at ∞ of H.

For the basics of *polyhedral* or, equivalently, *finitely generated cones* in $\mathbb{R}^r$ we refer to [Ger, Ps] or [StW]. Recall that a *frame* of a convex polyhedral cone is a finite set of halflines($\equiv$rays) the convex hull of which is the cone, and such that no halfline of the set is in the convex hull of the others. Note that each polyhedral cone has a frame, but the frame is, in general, not unique (e.g. if the cone is $\mathbb{R}^2$). The rays of a frame are called *generating rays($\equiv$edges)*. An *extreme halfline* or *ray* of a convex cone is a ray that is not a proper positive linear combination of any two rays of the cone. Any frame of a polyhedral cone contains all extreme rays of the cone, but the rays of a frame need not be extreme. A polyhedral cone $K \subset \mathbb{R}^n$ is *solid* if it has a nonvoid interior or, equivalently, the dimension of (the subspace spanned by) K is n. It is called *pointed* if $K \cap (-K) = \{0\}$. [Ger, Theorem 9] shows that if K is pointed, then it is generated by its extreme rays.

A polyhedral cone $K \subset \mathbb{R}^r$ is *system invariant* (with respect to (C, A, B)) if

$$C(K) \subset \mathbb{R}^l_+, \quad A(K) \subset K, \quad B(\mathbb{R}^k_+) \subset K.$$

Note the slight abuse of notation: here we treat the matrices C, A, B as linear operators. Finally, cone$[S]$ will denote the cone generated by the set $S \subset \mathbb{R}^r$.

2. Primitive scalar function with simple dominant pole

In this section we shall make use of some ideas of [ADFB]. As a preliminary result we need

Lemma 2.1. *Let J_0 be a Jordan matrix with all eigenvalues in the open unit disc in $\mathbb{C}$. For every $\eta > 0$ the following algorithm determines in a finite number of steps a positive integer $Q(\eta)$ such that $m \geq Q(\eta)$ implies $|J_0^m| < \eta$ (here $|\cdot|$ denotes the operator norm induced by the l^∞ vector norm).*

Proof. Let D and N denote the diagonal and the nilpotent parts of J_0, and let $N^q \neq 0$, $N^{q+1} = 0$. Then, for $m \geq q$ we have

$$J_0^m = \sum_{j=0}^{q} \binom{m}{j} D^{m-j} N^j.$$

For the operator norm $|\cdot|$ induced by the l^∞ vector norm then

$$|J_0^m| \leq \sum_{j=0}^{q} \binom{m}{j} |D^{m-j}||N|^j = \sum_{j=0}^{q} \binom{m}{j} r^{m-j},$$

where r denotes the spectral radius of J_0. For $m \geq 2q-1$ the right-hand side is not greater than $(q+1)r^{m-q}\binom{m}{q}$. This is less than $\eta > 0$ if and only if

$$m(m-1)\cdots(m-q+1) < \frac{q!r^{q-m}\eta}{q+1}.$$

The latter inequality is (for any log with base>1) implied by

$$m > q + \frac{q\log m + \log\frac{q+1}{q!\eta}}{|\log r|}.$$

For any nonnegative α and real β consider the equation

$$\alpha \log x + \beta = x.$$

The graphs of the functions on both sides show that the equation has at most two positive roots. Let $L(\alpha,\beta)$ denote the largest root, with the convention that $L(\alpha,\beta) := 0$ if there are no roots. Define

$$P(\eta) := L\left(\frac{q}{|\log r|}, \frac{\log\frac{q+1}{q!\eta}}{|\log r|} + q\right),$$

$$Q(\eta) := \max\{[P(\eta)], 2q-1\},$$

where $[x]$ denotes the smallest integer not less than x. By construction, then $m \geq Q(\eta)$ implies $|J_0^m| < \eta$, as wished. $\triangle$

The following theorem provides an upper estimate on the dimension of a nonnegative realization.

Theorem 2.2. *Let*

$$H(z) = \frac{p_1 z^{n-1} + \cdots + p_n}{z^n - q_1 z^{n-1} - \cdots - q_n}$$

be a primitive transfer function given by coprime polynomials with real coefficients, with all poles in the closed unit disc and such that 1 *is a pole of order* 1*. Apply the notation of the preceding lemma, and assume that the impulse response satisfies* $h_k \geq 0$ *for all* $k \geq 0$*. Assume also that* $\lim h_k = 1$ *(which is no essential restriction). Then* $H(z)$ *has a nonnegative realization of dimension not greater than*

$$Q\left(\frac{1}{(|h|K+2n-1)(2n-1)|T||T^{-1}|}\right) - 1 + (n+1)Q\left(\frac{1}{(|h|K+2n-1)|T||T^{-1}|}\right).$$

(The additional notation T, K, h *will be found in the course of the following proof, see, in particular, the formula* (1)*.)*

Proof. Let the triple (g, F, h), where

$$g = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ q_n & q_{n-1} & q_{n-2} & \dots & q_1 \end{pmatrix}, h = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{pmatrix}$$

be a minimal realization of the transfer function H (in the so called observability form, cf., e.g., [Sont, Sect. 6.6]). Let J be (a fixed) Jordan matrix of F with transformation matrix T such that $F = TJT^{-1}$. For every $m \in \mathbb{N}$ then we have

$$|F^m| \leq |T||J^m||T^{-1}|.$$

Here and in what follows $|F|$ denotes the matrix (operator) norm of F induced by the l^∞ vector norm $|\cdot|$, i.e. $|F| = \max_j \sigma_j$, where σ_j is the sum of the moduli of the entries in the jth row of F. Since F is a nonderogatory matrix, J contains only 1 Jordan block (of size s_k) for each eigenvalue z_k. By assumption, we have then

$$J = J(1,1) \oplus J(z_2, s_2) \oplus \cdots \oplus J(z_r, s_r),$$

where for each eigenvalue $z \in \mathbb{C}$ the matrix $J(z, s)$ is the (upper diagonal) Jordan block of size s. Hence we have

$$J(z,s)^m = \begin{pmatrix} z^m & \binom{m}{1} z^{m-1} & \dots & \binom{m}{s-1} z^{m-s+1} \\ 0 & z^m & \dots & \binom{m}{s-2} z^{m-s+2} \\ \dots\dots\dots\dots & & & \\ 0 & 0 & \dots & z^m \end{pmatrix} \text{ for } \quad m \geq s-1,$$

and there are only $m+1$ nonzero (consecutive) diagonals (upwards) for $m < s-1$. With the notation $b := |z| < 1$, in the latter case it follows that $|J(z, s)^m| = (1 + b)^m$, where $0 \leq b < 1$. Hence we have

$$|J(z,s)^m| \leq |J(z,s)^{s-1}| = (1+b)^{s-1} \qquad \text{for} \quad m \leq s-1.$$

On the other hand, for $m \geq s - 1$ we obtain

$$J(b,s,m) := |J(z,s)^m| = b^m + \binom{m}{1} b^{m-1} + \cdots + \binom{m}{s-1} b^{m-s+1}.$$

Hence

$$J(b,s,m+1) = b(m+1) \sum_{k=0}^{s-1} \binom{m}{k} b^{m-k} \frac{1}{m+1-k} \leq \frac{b(m+1)}{m+2-s} J(b,m,s).$$

If we have $\frac{b(m+1)}{m+2-s} \leq 1$ or, equivalently,

$$\text{if} \quad m \geq \frac{s-2+b}{1-b} =: \nu, \quad \text{then} \quad J(b,s,m+1) \leq J(b,s,m).$$

Denote the smallest nonnegative integer not less than ν by $\hat{\nu}$, and introduce

$$J(b,s) := \max\{J(b,s,m) : \ 1 \leq m \leq \max[\hat{\nu}, s-1]\}.$$

Then

$$J(b,s) = \max[(1+b)^{s-1}, \ J(b,s,k) : \ s \leq k \leq \hat{\nu}]$$

and, by what has been proved, for every nonnegative integer m we have

$$J(b,s,m) \leq J(b,s).$$

It follows that

$$|J^m| \leq \max\{J(b_j, s_j) : \ 1 \leq j \leq r\} =: K_0.$$

Hence we have for every nonnegative integer m the estimate

$$|F^m| \leq K_0 |T||T^{-1}| =: K \tag{1}$$

Let $J_0 := J(z_2, s_2) \oplus \cdots \oplus J(z_r, s_r)$. For every nonnegative integer m we have

$$\begin{aligned} |gT\{J^m - [J(1,1) \oplus [0]]\}T^{-1}h| \leq |gT|_1 |0 \oplus J_0^m||T^{-1}h| \leq \\ \leq |g|_1 |h||T||T^{-1}||J_0^m| \end{aligned} \tag{2}$$

Here $[0]$ denotes the square 0 matrix of size $s_2 + \cdots + s_r = n-1$, $|\cdot|_1$ denotes the l^1-norm, and all other norms are the l^∞ vector or induced matrix (operator) norms. Then $|g|_1 = 1$, and the right-hand side clearly tends to 0 as $m \to \infty$. More exactly, the right-hand side is smaller than a positive ε exactly when m is so large that

$$|J_0^m| < \frac{\varepsilon}{|h||T||T^{-1}|}. \tag{3}$$

In the notation of Lemma 2.1 , this is implied by

$$m \geq M(\varepsilon) := Q(\frac{\varepsilon}{|h||T|T^{-1}|}). \tag{4}$$

The left-hand side of (2) is clearly $|gF^m h - c| = |h_m - c|$, where $c \in \mathbb{R}$. This side tends also to 0, hence, by assumption, $c = 1$. We have obtained that (4) implies

$$|h_m - 1| < \varepsilon. \tag{5}$$

The spectra of the matrices F and J consist of the part $\{1\}$ and the (disjoint) nonunitary part $\{z_2, \ldots, z_r\}$. Denote the corresponding spectral projections by $P_k(F), P_k(J)$ $(k = 1, 0)$, respectively. Since $F = TJT^{-1}$, we clearly have

$$P_k(F) = TP_k(J)T^{-1}, \quad |P_k(J)| = 1, \quad |P_k(F)| \leq |T||T^{-1}| \quad (k = 0, 1).$$

Let $w \in \mathbb{R}^n$, $\eta > 0$. Consider the cube($\equiv$ball in the l^∞ norm) $B(w, \eta)$ defined by

$$B(w, \eta) := \{x \in \mathbb{R}^n : |x - w| \leq \eta\}.$$

The estimate (1) shows that for every $j \in \mathbb{N}$ we have $F^j B(0, \eta) \subset B(0, K\eta)$. Consider now the (column) vector

$$v := (1, 1, \ldots, 1)^T \in \mathbb{R}^n.$$

The form of the matrix F shows that $Fv = v$. Hence

$$F^j B(v, \eta) = F^j(v + B(0, \eta)) \subset v + B(0, K\eta) = B(v, K\eta).$$

The spectral projection $P_1(F)$ maps onto the 1-dimensional subspace generated by v. Further, the matrix $P_1(F)$ (in the used basis) has only real entries: indeed, it is $\lim_{t \to 1+0}(t-1)(t-F)^{-1}$. Hence for every $x \in \mathbb{R}^n$ there is a unique $p(x) \in \mathbb{R}$ such that

$$P_1(F)x = p(x)v.$$

Let $y_0 \in B(0, \eta)$. For $m \geq M(\eta)$ then

$$|F^m P_0(F) y_0| = |TJ_0^m P_0(J)T^{-1} y_0| \leq |T||T^{-1}||J_0^m|\eta < \frac{\eta^2}{|h|}.$$

Let $x \in B(v, \eta)$. Then $x_0 := x - v \in B(0, \eta)$, and

$$x = v + P_1(F)x_0 + P_0(F)x_0.$$

Therefore

$$v + p(x_0)v + F^m P_0(F)x_0 = F^m x \in B(v, K\eta).$$

The sequence $\{F^m P_0(F)x_0;\ m \in \mathbb{N}\}$ tends to 0 as $m \to \infty$, hence $[1 + p(x_0)]v \in B(v, K\eta)$.

Let $\mu > 0$. Choose $\eta \equiv \eta(\mu) > 0$ so small that

$$0 < \eta \leq \frac{|h|}{|h|K + \mu} \tag{6}$$

Then $|h|(1 - K\eta) \geq \eta\mu > 0$. Hence the numbers

$$d := 1 - K\eta, \qquad f := \frac{d\eta}{\mu} = \frac{(1 - K\eta)\eta}{\mu}$$

are positive, and we have $\frac{|h|d}{\mu} \geq \eta$. For $m \geq M(\eta)$ this implies

$$|F^m P_0(F)x_0| \leq \frac{\eta^2}{|h|} \leq \frac{d\eta}{\mu} = f.$$

Hence for every $x \in B(v, \eta(\mu))$ and $m \geq M(\eta(\mu))$ we have

$$F^m x \in B([1 + p(x_0)]v, f).$$

Note that 1 is the distance in l^∞ norm of v from each of the $n-1$-dimensional coordinate subspaces. Introduce the notation $a := [1 + p(x_0)]v \in B(v, K\eta)$. Then the coordinate(s) of a satisfy $1 + p(x_0) \geq 1 - K\eta = d > 0$. Hence $|a| \geq d$ and, with the notation

$$k_x := \frac{1}{1 + p(x_0)}, \qquad \text{we have} \qquad a = [1 + p(x_0)]v = \frac{1}{k_x} v.$$

Since $f = \frac{d\eta}{\mu} \leq \frac{|a|\eta}{\mu} = \frac{\eta}{\mu k_x}$, we have $B(a, f) \subset B(\frac{v}{k_x}, \frac{\eta}{\mu k_x})$. It follows that

$$B(a, f) \subset \text{cone}[B(\frac{v}{k_x}, \frac{\eta}{\mu k_x})] = \text{cone}[B(v, \frac{\eta}{\mu})].$$

Hence for $m \geq M(\eta(\mu))$ the elements of $F^m B(v, \eta)$ are all in $\text{cone}[B(v, \frac{\eta}{\mu})]$.

Note that for every positive number t the ball $B(0, t) \subset \mathbb{R}^n$ is contained in the n-dimensional simplex $S(0, [2n-1]t)$ defined as the convex hull of its following $n+1$ vertices:

$$(-t, \ldots, -t), ([2n-1]t, -t, \ldots, -t), \ldots, (-t, \ldots, -t, [2n-1]t).$$

Hence we clearly have

$$B(0, t) \subset S(0, [2n-1]t) \subset B(0, [2n-1]t).$$

If Y denotes B or S, introduce the notation $Y(v, t) := v + Y(0, t)$. Then

$$B(v, t) \subset S(v, [2n-1]t) \subset B(v, [2n-1]t).$$

An estimation of the number of the generating rays of a polyhedral cone G invariant with respect to (g, F, h) runs as follows. To the minimal realization (g, F, h) determine the numbers K_0, $|T|, |T^{-1}|, K$ by (1), as well as the (positive) norm $|h|$. Define $\mu := 2n - 1$. Choose

$$\eta := \frac{|h|}{|h|K + 2n - 1} > 0,$$

so that (6) holds, and then choose $M(\eta)$ as above. For $m \geq M(\eta)$ we then have

$$F^m S(v, \eta) \subset F^m B(v, \eta) \subset \text{cone}[B(v, \frac{\eta}{\mu})] \subset \text{cone}[S(v, \eta)].$$

Define the vectors $u_k := (h_k, h_{k+1}, \ldots, h_{k+n-1})^T \in \mathbb{R}^n$ for $k \in \mathbb{N}$. Let

$$\varepsilon := \frac{\eta}{\mu} \equiv \frac{\eta}{2n-1}.$$

Then, as we have seen (cf. (5)),

$$m \geq M(\varepsilon) \quad \text{implies} \quad |u_m - v| < \varepsilon,$$

hence $u_m \in B(v, \varepsilon)$. Since $Fu_k = u_{k+1}$ for every $k \in \mathbf{N}$, the images Fu_m for $m \geq M(\varepsilon)$ stay in $B(v,\varepsilon) \subset S(v,\eta)$. We have seen above that for $m \geq M(\eta)$ we have $F^m S(v,\eta) \subset \text{cone}[S(v,\eta)]$. Since the sum of polyhedral cones is polyhedral, it follows as in [ADFB, Theorem 4.1] that the polyhedral cone G defined by

$$G := \text{cone}[u_1, \ldots u_{M(\varepsilon)-1}, S(v,\eta), FS(v,\eta), \ldots F^{M(\eta)-1}S(v,\eta)] \subset \mathbb{R}^n_+$$

is invariant with respect to (g, F, h).

The number of the edges of any (polyhedral) cone$[F(B)]$ is not greater than the number of the edges of the polyhedral cone$[B]$. Hence the number of the generating rays of the cone G is at most

$$N := M(\varepsilon) - 1 + (n+1)M(\eta).$$

Note that $M(\varepsilon) \geq M(\eta)$ implies here

$$N \leq (n+2)M(\varepsilon) - 1.$$

Since

$$\varepsilon = \frac{|h|}{[|h|K + 2n - 1][2n-1]},$$

we have obtained the value

$$N = M(\frac{|h|}{[|h|K + 2n - 1][2n-1]}) - 1 + (n+1)M(\frac{|h|}{|h|K + 2n - 1}).$$

The theory of nonnegative realizations (see, e.g., [FN1, Prop. 2.2]) shows that there is such a realization of dimension not greater than N, and the proof is complete. △

3. The matrix-valued case

Assume now that the transfer function $H(z)$ is a rational $l \times k$ matrix-valued function which vanishes at infinity (MIMO case). Recall that [Far, Theorem 2.3] or [FN2, Theorem 2.4] (cf. also [BF]) shows that H has a nonnegative realization if and only if each entry of H has a nonnegative realization. Let (C, A, B) be a minimal realization of H and let $n := \text{dimspace}(A)$ be its dimension. Then the entry h_{rs} of H has the (in general not minimal) realization $(C_{r\cdot}, A, B_{\cdot s})$, where $r\cdot$ denotes rth row and $\cdot s$ denotes sth column. It is shown in the proof of [FN2, Theorem 2.4] that if the entry h_{rs} of H has a nonnegative realization, then there exists an (in general *not pointed and not solid*) polyhedral cone P_{rs} that is system invariant with respect to the triple $(C_{r\cdot}, A, B_{\cdot s})$, i.e. (with a slight abuse of notation)

$$C_{r\cdot}(P_{rs}) \subset \mathbb{R}_+, \quad A(P_{rs}) \subset P_{rs}, \quad B_{\cdot s} \in P_{rs}.$$

Let e_{rs} denote the (minimal) number of the generators of the polyhedral cone P_{rs}. Then there is a matrix M_{rs} of size $n \times e_{rs}$ such that

$$P_{rs} = \{M_{rs}x : x \in \mathbb{R}_+^{e_{rs}}\} = M_{rs}(\mathbb{R}_+^{e_{rs}}).$$

Recall that then the polyhedral cone $P := \cap_r [\sum_s P_{rs}]$ is system invariant with respect to the triple (C, A, B) ([FN2, Theorem 2.4]). Under these conditions we have the following

Theorem 3.1. *Assume that each entry h_{rs} has a nonnegative realization generated as above by the polyhedral cone P_{rs} having exactly e_{rs} generating rays. Let $e_r := \sum_s e_{rs}$, and let $[\cdot]$ denote the integer part of the number. Then the $l \times k$ matrix-valued transfer function H has a nonnegative realization of dimension not greater than*

$$\binom{\sum_{r=1}^{l} \binom{e_r}{[\frac{e_r}{2}]} + 2(n-1)l}{n-1}.$$

Proof. By the definition above, the columns of the matrix M_{rs} constitute a *frame* for the cone P_{rs} in the sense of [Ger, p.300]. By our assumption, the (minimal) number of the generating rays of the polyhedral A-invariant cone P_{rs} is e_{rs}. The cone $P_r := \sum_s P_{rs}$ is also polyhedral, and the number of its generating rays is $\leq \sum_s e_{rs} = e_r$ (cf. [Ps, Theor. I.4.11]).

Recall that the dual cone K^* of any cone $K \subset \mathbb{R}^n$ is defined by

$$K^* := \{x \in \mathbb{R}^n : (x, K) \subset \mathbb{R}_+\}.$$

(Here $(\cdot, \cdot)$ denotes scalar product in $\mathbb{R}^n$.) It is also known that for any polyhedral cone K the number $e(K)$ of its generating rays and the minimal number $i(K)$ of the linear homogeneous inequalities ($\equiv$halfspaces) defining K are connected in the following way (see, e.g., [StW, (2.8.3), (2.8.4), p.55] or [Ps, Theor. I.4.7-10]):

$$i(K^*) = e(K), \qquad e(K^*) = i(K).$$

Further, [Ger, Theorem 11] (a variant of the famous Minkowski-Weyl theorems) shows that any *solid* polyhedral cone K in $\mathbb{R}^n$ is the intersection of a finite number of halfspaces such that each hyperplane bounding the corresponding halfspace contains exactly $n-1$ linearly independent elements of K and is spanned by rays of any frame of K. Hence different bounding hyperplanes are spanned by different $(n-1)$ element subsets of the whole set E of the $e(K)$ generating rays of the frame. It follows that the minimal number $i(K)$ of the needed halfspaces is $\leq$ the number of such subsets, i.e.

$$i(K) \leq \binom{e(K)}{n-1}.$$

Note that the original work by Minkowski [Mink, pp.40-43] essentially contains a (dual) estimate

$$e(K) \leq \binom{i(K)}{n-1}$$

under the condition that the number of the linearly independent hyperplanes ($\equiv$linear functionals) defining K is n.

For a *not necessarily solid* polyhedral cone K consider the dimension $d \geq 1$ of the smallest subspace of $\mathbb{R}^n$ containing K. Then K is solid in the relative topology of that subspace. Further, the subspace itself can be described by $n-d$ linear *equalities*, hence by $2(n-d)$ *inequalities*. Therefore in this general case we obtain

$$i(K) \leq \binom{e(K)}{d-1} + 2(n-d).$$

A simple (consequently, unsharp) estimate, independent of d, is then for every nonzero polyhedral cone $K \subset \mathbb{R}^n$

$$i(K) \leq \binom{e(K)}{[\frac{e(K)}{2}]} + 2(n-1).$$

(Here $[\cdot]$ denotes the integer part of the number.) It follows that

$$e(K) = i(K^*) \leq \binom{e(K^*)}{[\frac{e(K^*)}{2}]} + 2(n-1) = \binom{i(K)}{[\frac{i(K)}{2}]} + 2(n-1).$$

Returning to our studied case, $e(P_r) \leq e_r$ implies that

$$i(P_r) \leq \binom{e_r}{[\frac{e_r}{2}]} + 2(n-1).$$

For the polyhedral cone $P = \cap_{r=1}^{l} P_r$ we obtain then

$$i(P) \leq \sum_{r=1}^{l} \binom{e_r}{[\frac{e_r}{2}]} + 2(n-1)l.$$

Since the polyhedral cone $P = \cap_r P_r$ is system-invariant for the minimal triple (C, A, B), by [FN1, remark after Theorem 2.4] it is *pointed* and *solid*, and we can apply the above inequality of Minkowski. Indeed, it is well-known (cf. [StW, Theorem 2.10.7]) that the dual polyhedral cone

$P^* \subset \mathbb{R}^n$ is also pointed and solid. Hence it has at least n generating rays, and can be given as

$$P^* = Q_{n\times N}(\mathbb{R}_+^N) \qquad (N \geq n),$$

where the subscript denotes the size of the matrix Q, and $\mathrm{rank}(Q) = n$. It is known that the dual P^{**} is the cone P and is given by

$$P = P^{**} = \{x \in \mathbb{R}^n : Q^T_{N\times n} x \geq 0\}.$$

Since $\mathrm{rank}(Q^T) = n$, there are n linearly independent inequalities among those determining the cone P. Applying the Minkowski inequality, we obtain

$$e(P) \leq \binom{i(P)}{n-1} \leq \binom{\sum_{r=1}^{l} \binom{e_r}{[\frac{e_r}{2}]} + 2(n-1)l}{n-1}.$$

It is well-known (cf. [FN1, Prop. 2.2]) that under our conditions there is a nonnegative realization of H of dimension not greater than $e(P)$. The proof is complete. △

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