Differential Equations Exercises

1 Separable DE

Solve the following differential equations and IVP's:

$$xy + y\ln(y)y' = 0$$
$$x^4y' = y^3$$
$$x^2y^2y' + 1 = y \quad y(1) = 2$$
$$y'\cot x + y = 2 \quad y(0) = -1$$

2 Linear DE

Solve the following linear ODEs, then compute the solution of the IVP by successive approximation.

$$y' = 2y + x + 1 \quad y(0) = -2$$
$$y' = x - y \quad y(0) = 1$$
$$y + y' + 1 = x^3 \quad y(0) = 2$$
$$y' + 4y = 2 \quad y(0) = -1$$

Solve the following linear ODEs and check the result by the Taylor series method or by the method of undetermined coefficients.

$$y' = 2y + x \quad y(1) = 2$$
$$y' = x - y + 3 \quad y(-1) = 2$$
$$y' + 1 = y + x^3 \quad y(1) = 2$$
$$y' + 4y = 2 \quad y(2) = -1$$

3 Exact DE and integrating factors

Determine whether the given DE is exact. If not, find an integrating factor (a multiplier). Give the general solution.

$$(2x + 3x^2y)\ \mathrm{d}x + (x^3 - 3y^2)\ \mathrm{d}y = 0$$

$$3\cos(3x - y)\ \mathrm{d}x - \cos(3x - y)\ \mathrm{d}y = 0$$

$$y\ \mathrm{d}x + (xy^3 + x\ln x)\ \mathrm{d}y = 0 \quad z = x$$

$$xy + y\ln yy' = 0$$

$$\frac{y}{x^2 + y^2}\ \mathrm{d}x - \frac{x}{x^2 + y^2}\ \mathrm{d}y = 0$$

$$(xy + y^4)\ \mathrm{d}x + (x^2 - xy^3)\ \mathrm{d}y = 0 \quad z = xy$$

$$2x\ \mathrm{d}x + (x - y)\ \mathrm{d}y = 0 \quad z = x + y$$

4 Mixed exercises to discuss before the first test

$$
\begin{aligned}
2x\ \mathrm{d}y + (x^2y^4+1)y\ \mathrm{d}x &= 0\\
y + (2xy+1)y' &= 0\\
(\sin^2 y + x\cot y)y' &= 1\\
5(1+x^2)y' &= 2xy + \frac{(1+x^2)^2}{y^4}\\
y' &= y^4\cos x + y\tan x\\
y' + y^2 &= \frac{2}{x^2}\\
x^3y' + x^2y - y^2 &= 2x^4\\
y' &= (y-x)^2+1\\
y' &= 2\frac{y}{x} + y^2 - x^2\\
y' &= -\frac{4}{x^2} - \frac{1}{x}y + y^2\\
(1+2y)\ \mathrm{d}x + (4-x^2)\ \mathrm{d}y &= 0\\
(xy'-1)\ln x &= 2y\\
xy' + (x+1)y &= 3x^2e^{-x}\\
x^2y' - \cos 2y = 1, &\quad y(+\infty) = \frac{9}{4}\pi\\
(2x^2-y^3)\ \mathrm{d}x - 3xy^2(1+\ln x)\ \mathrm{d}y &= 0
\end{aligned}
$$

Let us approximate the solution to the IVP

$$
\left.\begin{aligned} y' &= -4y \\ y(-1) &= 2 \end{aligned}\right\}
$$

at $x = 2$ by the explicit and the implicit Euler's method with constant step size $h = \frac{1}{2}$, and determine whether they are greater or smaller than the exact value.

5 2nd order linear ODE

Find the general solution to the differential equations by two different methods. Convert them to linear systems and solve the resulting systems. Compare the solutions.

$$\begin{aligned} y'' - 6y' + 13y &= 5e^{4x} \\ y'' - 6y' + 13y &= 5e^{4x}\cos 2x \\ y'' - 6y' + 13y &= 5e^{3x} \\ y'' - 6y' + 13y &= 5e^{3x}\cos 2x \end{aligned}$$

6 Direction fields and 1D phase portraits

Draw the direction field of the given equations and the 1D phase portrait (if it exists). Then sketch in some integral curves, using the information provided by the direction field.

$$\begin{aligned} y' &= x + y \\ y' &= -y^2 \\ y' &= x^2 + y^2 - 1 \\ y' &= (y-3)(y-1)(y+1) \\ y' &= \cos y \\ y' &= \cot \frac{y}{2}. \end{aligned}$$

7 First-order homogeneous linear systems and 2D linear phase portraits

Solve the following linear system and draw its phase portrait:

$$\begin{aligned} \dot{x} &= 3x + 6y, \\ \dot{y} &= -2x - 6y. \end{aligned}$$

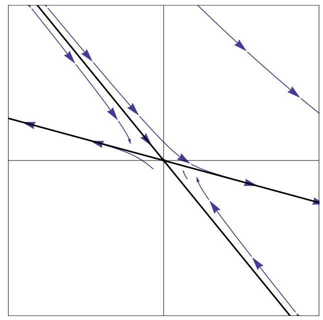

Figure 1: Saddle

Eigenvalues are $-\frac{3}{2}-\frac{\sqrt{33}}{2}$ and $\frac{\sqrt{33}}{2}-\frac{3}{2}$ hence the origin is a saddle. Eigenvectors are $(\frac{\sqrt{33}}{4}-\frac{9}{4},1)$ and $(-\frac{9}{4}-\frac{\sqrt{33}}{4},1)$ hence the solution is

$$c_1 e^{\left(-\frac{3}{2}-\frac{\sqrt{33}}{2}\right)t}\begin{pmatrix} -\frac{9}{4}+\frac{\sqrt{33}}{4} \\ 1 \end{pmatrix} + c_2 e^{\left(\frac{\sqrt{33}}{2}-\frac{3}{2}\right)t}\begin{pmatrix} -\frac{9}{4}-\frac{\sqrt{33}}{4} \\ 1 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = 9x - 6y,$$
$$\dot{y} = -4x + 7y.$$

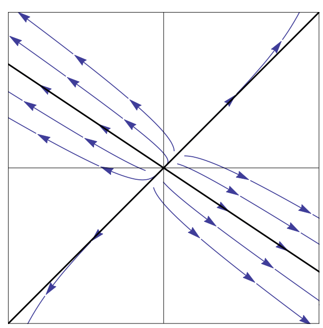

Figure 2: Unstable node

Eigenvalues are 13 and 3 hence the origin is an unstable node. Eigenvectors are $(-3,2)$ and $(1,1)$ hence the solution is

$$c_1 e^{13t}\begin{pmatrix} -3 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = -9x - 4y,$$
$$\dot{y} = -9x - y.$$

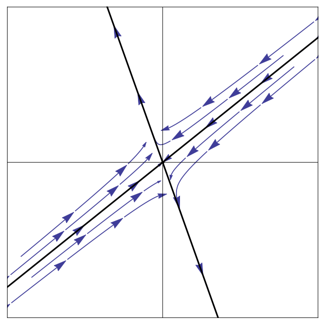

Figure 3: Saddle

Eigenvalues are $-5 - 2\sqrt{13}$ and $2\sqrt{13} - 5$ hence the origin is a saddle. Eigenvectors are $(\frac{4}{9} + \frac{2\sqrt{13}}{9}, 1)$ and $(\frac{4}{9} - \frac{2\sqrt{13}}{9}, 1)$ hence the solution is

$$c_1 e^{\left(-5-2\sqrt{13}\right)t} \begin{pmatrix} \frac{4}{9} + \frac{2\sqrt{13}}{9} \\ 1 \end{pmatrix} + c_2 e^{\left(2\sqrt{13}-5\right)t} \begin{pmatrix} \frac{4}{9} - \frac{2\sqrt{13}}{9} \\ 1 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = x - 8y,$$
$$\dot{y} = 2x - y.$$

Eigenvalues are $i\sqrt{15}$ and $-i\sqrt{15}$ hence the origin isa center. Eigenvectors are $(\frac{1}{2} + \frac{i\sqrt{15}}{2}, 1)$ and $(\frac{1}{2} - \frac{i\sqrt{15}}{2}, 1)$ hence the solution is

$$c_1 e^{i\sqrt{15}t} \begin{pmatrix} \frac{1}{2} + \frac{i\sqrt{15}}{2} \\ 1 \end{pmatrix} + c_2 e^{-i\sqrt{15}t} \begin{pmatrix} \frac{1}{2} - \frac{i\sqrt{15}}{2} \\ 1 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = 5x - 2y,$$
$$\dot{y} = 2x + 5y.$$

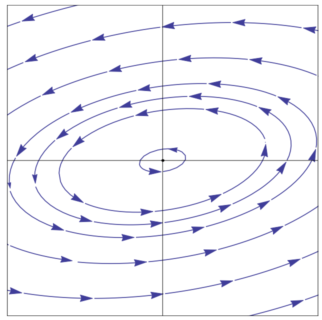

Figure 4: Center

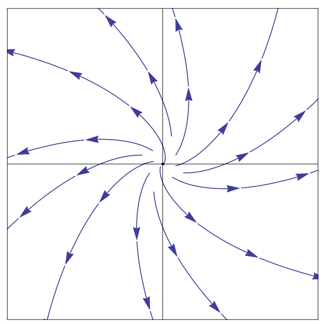

Figure 5: Unstable spiral

Eigenvalues are $5+2i$ and $5-2i$ hence the origin is an unstable spiral. Eigenvectors are $(i,1)$ and $(-i,1)$ hence the solution is

$$c_1 e^{(5+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{(5-2i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = 9x - 8y,$$
$$\dot{y} = 9x.$$

Eigenvalues are $\frac{9}{2}+\frac{3i\sqrt{23}}{2}$ and $\frac{9}{2}-\frac{3i\sqrt{23}}{2}$ hence the origin is an unstable spiral. Eigenvectors are $(\frac{1}{2}+\frac{i\sqrt{23}}{6},1)$ and $(\frac{1}{2}-\frac{i\sqrt{23}}{6},1)$ hence the solution is

$$c_1 e^{\left(\frac{9}{2}+\frac{3i\sqrt{23}}{2}\right)t}\begin{pmatrix} \frac{1}{2}+\frac{i\sqrt{23}}{6} \\ 1 \end{pmatrix} + c_2 e^{\left(\frac{9}{2}-\frac{3i\sqrt{23}}{2}\right)t}\begin{pmatrix} \frac{1}{2}-\frac{i\sqrt{23}}{6} \\ 1 \end{pmatrix}.$$

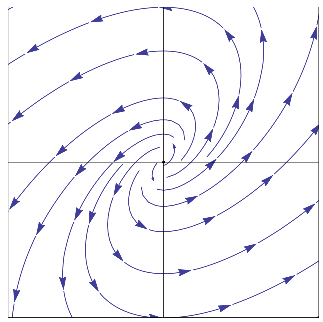

Figure 6: Unstable spiral

Solve the following linear system and draw its phase portrait:

$$\dot{x} = -3x - 5y,$$
$$\dot{y} = 2x - 4y.$$

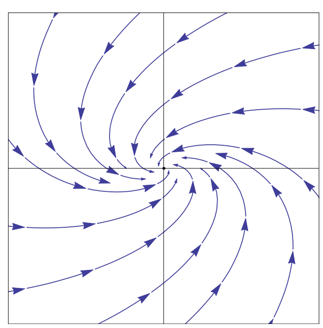

Figure 7: Stable spiral

Eigenvalues are $-\frac{7}{2}+\frac{i\sqrt{39}}{2}$ and $-\frac{7}{2}-\frac{i\sqrt{39}}{2}$ hence the origin is a stable spiral. Eigenvectors are $(\frac{1}{4}+\frac{i\sqrt{39}}{4}, 1)$ and $(\frac{1}{4}-\frac{i\sqrt{39}}{4}, 1)$ hence the solution is

$$c_1 e^{\left(-\frac{7}{2}+\frac{i\sqrt{39}}{2}\right)t} \begin{pmatrix} \frac{1}{4}+\frac{i\sqrt{39}}{4} \\ 1 \end{pmatrix} + c_2 e^{\left(-\frac{7}{2}-\frac{i\sqrt{39}}{2}\right)t} \begin{pmatrix} \frac{1}{4}-\frac{i\sqrt{39}}{4} \\ 1 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = -3x - 8y,$$
$$\dot{y} = -4x - y.$$

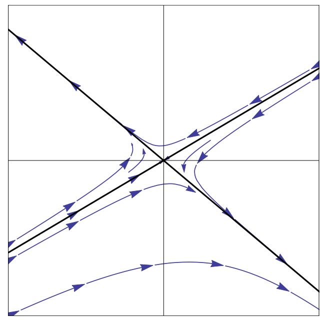

Figure 8: Saddle

Eigenvalues are $-2-\sqrt{33}$ and $\sqrt{33}-2$ hence the origin is a saddle. Eigenvectors are $(\frac{1}{4}+\frac{\sqrt{33}}{4},1)$ and $(\frac{1}{4}-\frac{\sqrt{33}}{4},1)$ hence the solution is

$$c_1 e^{\left(-2-\sqrt{33}\right)t}\begin{pmatrix} \frac{1}{4}+\frac{\sqrt{33}}{4} \\ 1 \end{pmatrix} + c_2 e^{\left(\sqrt{33}-2\right)t}\begin{pmatrix} \frac{1}{4}-\frac{\sqrt{33}}{4} \\ 1 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = -3x + 5y,$$
$$\dot{y} = -4x + 9y.$$

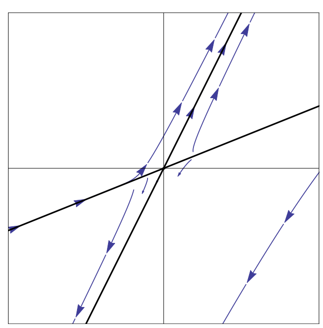

Figure 9: Saddle

Eigenvalues are 7 and -1 hence the origin is a saddle. Eigenvectors are $(1,2)$ and $(5,2)$ hence the solution is

$$c_1 e^{7t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Solve the following linear system and draw its phase portrait:

$$\dot{x} = 6x + 3y,$$
$$\dot{y} = 4x + y.$$

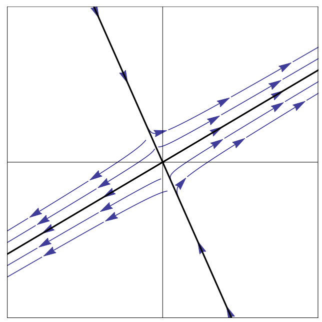

Figure 10: Saddle

Eigenvalues are $\frac{7}{2} + \frac{\sqrt{73}}{2}$ and $\frac{7}{2} - \frac{\sqrt{73}}{2}$ hence the origin is a saddle. Eigenvectors are $(\frac{5}{8} + \frac{\sqrt{73}}{8}, 1)$ and $(\frac{5}{8} - \frac{\sqrt{73}}{8}, 1)$ hence the solution is

$$c_1 e^{\left(\frac{7}{2}+\frac{\sqrt{73}}{2}\right)t} \begin{pmatrix} \frac{5}{8} + \frac{\sqrt{73}}{8} \\ 1 \end{pmatrix} + c_2 e^{\left(\frac{7}{2}-\frac{\sqrt{73}}{2}\right)t} \begin{pmatrix} \frac{5}{8} - \frac{\sqrt{73}}{8} \\ 1 \end{pmatrix}.$$

8 First-order inhomogeneous linear systems

Solve the following linear systems:

$$\dot{x} = 6x + 3y + e^t,$$
$$\dot{y} = 4x + y + e^{2t}.$$

$$\dot{x} = 6x + 3y - e^t,$$
$$\dot{y} = 4x + y + 4e^{2t}.$$

9 Linearization

Find the equilibrium points and determine their stability. In 2D determine also the type of the linear part.

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\sin x - 3y.\end{aligned}$$

$$\begin{aligned}\dot{x} &= y^2 - 1,\\ \dot{y} &= x^2 + y^2 - 2.\end{aligned}$$

$$\begin{aligned}\dot{x} &= x^2 + y^2 - 25,\\ \dot{y} &= xy - 12.\end{aligned}$$

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x^3 - x + xy.\end{aligned}$$

$$\begin{aligned}\dot{x} &= y - x^2 - x,\\ \dot{y} &= 3x - x^2 - y.\end{aligned}$$

$$\begin{aligned}\dot{x} &= (x-1)(y-1),\\ \dot{y} &= xy - 2.\end{aligned}$$

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \sin(x+y).\end{aligned}$$

$$\begin{aligned}\dot{x} &= \ln(y^2 - x),\\ \dot{y} &= x - y - 1.\end{aligned}$$

$$\begin{aligned}\dot{x} &= 4y^2 - x^2,\\ \dot{y} &= 2xy - 4y - 8.\end{aligned}$$

$$\dot{x} = 2y,$$
$$\dot{y} = x^2 - y^3 - 1.$$

$$\dot{x} = x - y,$$
$$\dot{y} = x^2 + y^2 - 2.$$

$$\dot{x} = x + y + 1,$$
$$\dot{y} = y + \sqrt{1 + 2x^2}.$$

$$\dot{x} = xy - 2,$$
$$\dot{y} = (2x - y)(x - 2).$$

10 Lyapunov-functions

Determine the stability of the origin by an appropriate Lyapunov-function:

$$\dot{x} = x^3 - y,$$
$$\dot{y} = x + y^3.$$

$$\dot{x} = y - x + xy,$$
$$\dot{y} = x - y - x^2 - y^3.$$

$$\dot{x} = 2y^3 - x^5,$$
$$\dot{y} = -x - y^3 + y^5.$$

$$\dot{x} = xy - x^3 + y^3,$$
$$\dot{y} = x^2 - y^3.$$

$$\dot{x} = y - 3x - x^3,$$
$$\dot{y} = 6x - 2y.$$

$$\dot{x} = 2y - x - y^3,$$
$$\dot{y} = x - 2y.$$

$$\dot{x} = x - y - x^3,$$
$$\dot{y} = x + y + y^3.$$

$$\dot{x} = xy^2 - x^3,$$
$$\dot{y} = -y^3 - 2x^2 y.$$

$$\dot{x} = -xy^4,$$
$$\dot{y} = x^6 y.$$

$$\dot{x} = xy + x^3,$$
$$\dot{y} = -y + y^2 - x^3 + x^4.$$

11 Laplace's transform

Solve the following ODE's (or systems of ODE's) by Laplace's transform:

$$y'' - y' - 12y = 3x - 2,$$
$$y(0) = y'(0) = 0.$$

$$y'' - 2y' + y = 6xe^x,$$
$$y(0) = 2, y'(0) = 1.$$

$$y'' - 4y' + y = 8\sin 2x,$$
$$y(0) = 2, y'(0) = 4.$$

$$\dot{x} = 2x + e^{-t},$$
$$\dot{y} = x - y + e^t,$$
$$x(0) = y(0) = 0.$$