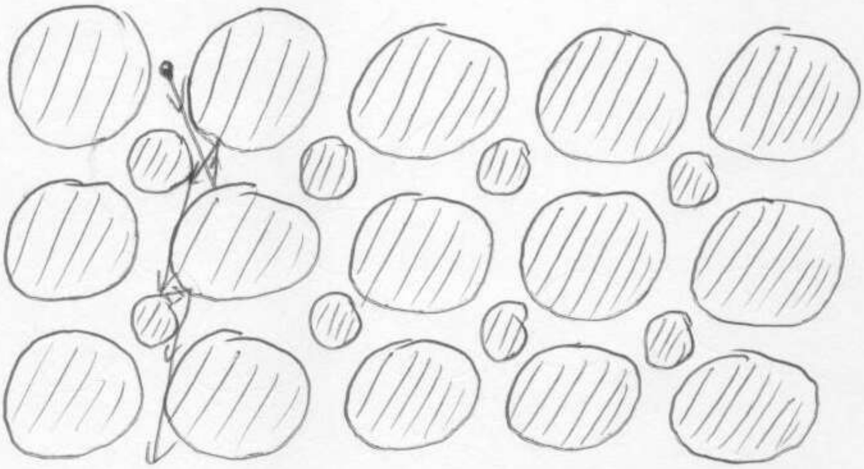
Decay of correlations and singularities in billiards

A ball is moving in an infinite, periodic pinball machine:



Let $X(t) \in \mathbb{R}^2$ be the position of the ball after time t.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be an „observable" quantity

[for simplicity, assume that f is also periodic, just like the obstacles.]

Then $t \mapsto f(X(t))$ is of coure a deterministic function: it is determined completely by the initial state of the ball (meaning: position + velocity).

<u>BUT</u>, if the <u>intitial</u> <u>state</u> is <u>random</u>, then $f(X(t))$ is also random, so it is a

$\boxed{\text{stochastic process}}$.

Important: This process is very much non-Markovian, with very long memory: everything is determined at the very beginning.

2/4

So it is a $\boxed{\text{shocking fact}}$:

Exponential decay of correlations: $\exists\, C<\infty,\ \lambda>0$ s.t.

$$\boxed{r\big(f(X(0)), f(X(t))\big) \le C\, e^{-\lambda t}} \quad \forall t>0 \qquad (*)$$

where $r(A,B) := \dfrac{\mathrm{Cov}(A,B)}{\sqrt{\mathrm{Var}\,A}\,\sqrt{\mathrm{Var}\,B}}$ is the correlation coefficient of random variables A, B.

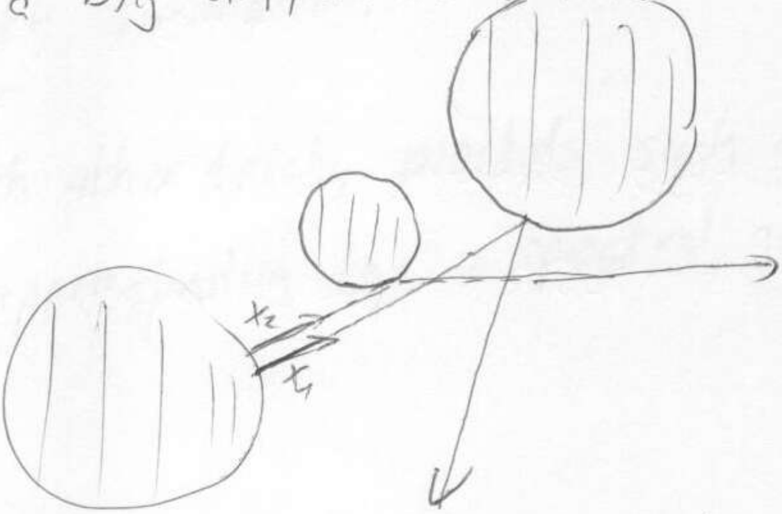
[at least for a wide class of observables f and initial distributions]

That is: this deterministic system exhibits as nice mixing properties as the most well-behaved Markov stochastic processes.

Catchword: Ther is CHAOS in the system: the function $X(t)$ is very sensitive to the choice of $X(0)$.

Bad news: We can only prove exponential decay of correlations in $\boxed{\text{2 dimensions}}$ (as in the drawing), not in 3D.

The difficulty is caused by $\boxed{\text{singularities}}$:
it makes a big difference whether



an obstacle is just barely hit or barely missed,

$\boxed{\text{AT LEAST FOR THE PROOFS THAT PRESENTLY EXIST}}$

Task: Try to understand wherther (or not) singularities have a real role in correlation decay: Is the exponent λ in the formula $(*)$ bigger or smaller in a system which has many singularities?

Solution plan:

Try to determine λ numerically as precisely as we can, for different configurations of the obstacles. 4/4

This can be done either

- with simulation,

  or

- with other tricky methods, such as approximating the „spectral gap".