Mathematical methods of statistical physics
Homework

Exercise 1. Let $(\Omega, \mathcal{F})$ be a measure space and X an index set. Let $T : \Omega \to X$, and

$$A_x = \{\omega \in \Omega \mid T(\omega) = x\}.$$

Let k be a kernel from X to Ω such that for every $x \in X$, $k(x, \cdot)$ is concentrated on A_x. Let μ be a measure on Ω and π be a measure on X such that $\mu = \pi * k$. Prove that k is a probability kernel if and only if $\pi = T_* \mu$.

Exercise 2. Let $\Omega = \mathbb{R}^2$, $\mu = \mathrm{Leb}_{\mathbb{R}^2}$, $X = \mathbb{R}$, $T(x, y) = \frac{1}{2m}(x^2 + y^2) = E$, $\pi = \mathrm{Leb}$. Let $\mu = \pi * k$.

a.) Show that $k(E, \cdot) = \mathrm{const} \cdot \mathrm{arclength}$.

b.) Find the "constant" $\mathrm{const} = \mathrm{const}(E)$.

c.) compare withthe case $T = \sqrt{x^2 + y^2}$.

Exercise 3. Let $X_1, X_2, \ldots$ be a Markov chain on the state space $\{1, 2, \ldots, K\}$ with initial distribution π and transition matrix $(p_{ij})_{i,j=1}^K$. Prove that for the entropy of the joint distribution of X_1 and X_2

$$S((X_1, X_2)) = S(\pi) + \sum_{i=1}^{K} S(p_{i\cdot})\pi_i$$

holds, where $p_{i\cdot}$ is row i of the transition matrix.

Exercise 4. Find the random variable $X : A \to \mathbb{R}$ with distribution μ with maximal relative entropy with respect to ν in the following scenarios. (See HW 4.6 in[HW] for details.)

a.) $A = \{1, 2, \ldots, K\}$, $\nu =$ counting measure,

b.) $A = [0, k]$, $\nu =$ Lebesgue,

c.) $A = \mathbb{R}$, $\nu =$ Lebesgue,

d.) $A = \mathbb{R}$, $\nu =$ Lebesgue, $\mathbb{E}X = 1$,

e.) $A = \mathbb{R}^+$, $\nu =$ Lebesgue, $\mathbb{E}X = 1$,

f.) $A = \mathbb{R}$, $\nu =$ Lebesgue, $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$,

g.) $A = \mathbb{N}$, $\nu =$ counting measure, $\mathbb{E}X = 2$.

Exercise 5. Consider the free gas. (For details, see Exercies 4.7, 5.4 and 5.5 in [HW].)

1. *Microcanonical setting:* Calculate the microcanonical partition function $Z(V, N, E)$, the microcanonical entropy $S(V, N, E)$, the temperature $T(V, N, E)$, the pressure $P(V, N, E)$ and the chemical potential $\mu(V, N, E)$.

2. *Canonical setting:* Calculate the free energy $A(V, N, \beta)$, the canonical pressure $P(V, N, \beta)$ and the canonical chemical potential $\mu(V, N, \beta)$.

3. *Grand canonical setting:* Calculate the grand canonical partition function $Z(V, \beta, \beta')$, the entropy density $s(V, N, E)$, the free energy per particle $a(V, N, \beta)$ and the grand free energy density $g(V, \beta, \beta')$.

Most importantly: find calculate the thermodynamic limits as $V \to \infty$ in all three cases, and compare the results.

Exercise 6. In what sense is the grand canonical partition function $Z(V, \beta, \beta')$ a moment generating function? Hint: calculate

$$
\begin{aligned}
M_H(\lambda) = \mathbb{E}e^{\lambda H} &= \int_{\Omega_1} e^{\lambda H(\omega)} \, \mathrm{d}\mu^{gr}_{V,\beta,\beta'}(\omega), \\
M_N(\lambda') = \mathbb{E}e^{\lambda' N} &= \int_{\Omega_2} e^{\lambda' N(\omega)} \, \mathrm{d}\mu^{gr}_{V,\beta,\beta'}(\omega), \\
M_{H,N}(\lambda) = \mathbb{E}e^{\lambda H + \lambda' N} &= \int_{\Omega_1} e^{\lambda H(\omega) + \lambda' N(\omega)} \, \mathrm{d}\mu^{gr}_{V,\beta,\beta'}(\omega).
\end{aligned}
$$

Exercise 7. Consider the configuration gas with Hamiltonian

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i<j} \Phi(|q_i - q_j|),$$

where Φ is tempered and stable. Let Λ_k be a square with side length $2^k - R$, $V_k = \mathrm{Leb}(\Lambda_k)$ and $N_k = \rho V_k$.

(a) Let

$$\tilde{a}(\rho, \beta) = \lim_{k \to \infty} \frac{1}{V_k} \log Z_k(V_k, N_k, \beta).$$

Show that $\tilde{a}$ is midpoint concave in ρ. That is, show if $\rho = \frac{\rho_1 + \rho_2}{2}$, then

$$\tilde{a}(\rho) \geq \frac{\tilde{a}(\rho_1) + \tilde{a}(\rho_2)}{2}.$$

(b) Let $\hat{\Lambda}_k$, $k \to \infty$ be an arbitrary growing sequence of boxes, $\hat{V}_k = \mathrm{Leb}(\hat{\Lambda}_k)$ and $\hat{N}_k = \rho \hat{V}_k$. Show that

$$-\lim_{k \to \infty} \frac{1}{\hat{V}_k \beta} \log \hat{Z}_k(\hat{\Lambda}_k, \hat{N}_k, \beta) = -\lim_{k \to \infty} \frac{1}{V_k \beta} \log Z_k(\Lambda_k, N_k, \beta) = a(\rho, \beta).$$

Exercise 8. Consider the 1D Ising model with L sites. Denote the partition function by $Z_L(\beta, h)$. (See Exercise 10.6 in[HW] for details.)

(a) Consider periodic boundary conditions. Show that $Z_L(\beta, h) = \mathrm{Tr}(T^L)$, where

$$T = \begin{bmatrix} e^{\beta(1+h)} & e^{-\beta} \\ e^{-\beta} & e^{\beta(1-h)} \end{bmatrix}$$

(b) Calculate the eigenvalues of T.

(c) Find a matrix power formula for $Z_L(\beta, h)$ for open boundary conditions.

Exercise 9. Consider the 2D Ising model. Let $P_\Lambda(\beta, h)$ be the pressure of the system in a box Λ. Let $A_m \subset \mathbb{Z}^2$ be a box of size m. Let $A_m \to \mathbb{Z}^2$ in the sense of van Hove. Show that $\lim_{m\to\infty} P_{A_m}(\beta, h) = \lim_{m\to\infty} P_{\Lambda_m}(\beta, h)$, where Λ_m is a square of side length 2^m.

Exercise 10. List of other relevant exercises from [HW]: 2.8, 4.1, 4.2, 4.3, 5.1, 5.2, 5.3, 10.1, 10.2, 10.3, 10.4, 11.2, 11.5, 12.2

References

[HW] Tóth Imre Péter: homework sheets for Mathematical Statistical Physics – with solutions to many exercises.
`http://math.bme.hu/~mogy/oktatas/stat_fiz_mat_modszerei/math_stat_phys/exercises/`
or find the link from `http://math.bme.hu/~mogy/oktatas.html`