Mathematical Statistical Physics – LMU München, summer semester 2012
Hartmut Ruhl, Imre Péter Tóth
Homework sheet 2 – due on 04.05.2012 – and exercises for the class on 27.04.2012

2.1 *Continuity of the measure*

(a) Prove the following:

Theorem 1 *(Continuity of the measure)*

i. *If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $A_1, A_2, \dots$ is an increasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \subset A_{i+1}$ for all i), then $\mu(\cup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).*

ii. *If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $A_1, A_2, \dots$ is a decreasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \supset A_{i+1}$ for all i) and $\mu(A_1) < \infty$, then $\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).*

(b) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

2.2 *Usefulness of the linearity of the expectation.* A building has 10 floors, not including the ground floor. On the ground floor, 10 people get into the elevator, and every one of them chooses a destination at random, uniformly out of the 10 floors, independently of the others. Let X denote the number of floors *on which the elevator stops* – i.e. the number of floors that were chosen by at least one person. Calculate the expectation and the variance of X. *(hint: First notice that the distribution of X is hard to calculate. Find a way to calculate the expectation and the variance without that.)*

2.3 **(homework)** Calculate the characteristic function of the normal distribution $\mathcal{N}(m, \sigma^2)$. (Remember the definition from the old times: $\mathcal{N}(m, \sigma^2)$ is the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgues measure)

$$f_{m,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

You can save yourself some paperwork if you only do the calculation for $\mathcal{N}(0, 1)$ and reduce the general case to this using the relation between different normal distributions. You can and should use the fact that

$$\int_{-\infty}^{\infty} f_{m,\sigma^2}(x)\,\mathrm{d}x = 1$$

for every m and σ.

2.4 *Dominated convergence and continuous differentiability of the characteristic function.*
The Lebesgue dominated convergence theorem is the following

Theorem 2 (dominated convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \dots$ measurable real valued functions on Ω which converge to the limit function pointwise, μ-almost everywehere. (That is, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$, except possibly for a set of x-es with μ-measure zero.) Assume furthermore that the f_n admit a common* integrable dominating function*: there exists a $g : \Omega \to \mathbb{R}$ such that $|f_n(x)| \le g(x)$ for every $x \in \Omega$ and $n \in \mathbb{N}$, and $\int_\Omega g\,\mathrm{d}\mu < \infty$. Then (all the f_n and also f are integrable and)*

$$\lim_{n\to\infty} \int_\Omega f_n\,\mathrm{d}\mu = \int_\Omega f\,\mathrm{d}\mu.$$

Use this theorem to prove the following

Theorem 3 (differentiability of the characteristic function) *Let X be a real valued random variable, $\psi(t) = \mathbb{E}(e^{itX})$ its characteristic function and $n \in \mathbb{N}$. If the n-th moment of X exists and is finite (i.e. $\mathbb{E}(|X|^n) < \infty$), then ψ is n times continuously differentiable and*

$$\psi^{(k)}(0) = i^k \mathbb{E}(X^k), \quad k = 0, 1, 2, \ldots, n.$$

2.5 *Weak convergence and densities.*

(a) **(homework)** Prove the following

Theorem 4 *Let $\mu_1, \mu_2, \ldots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \ldots$ and f, respectively. Suppose that $f_n(x) \stackrel{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \ldots$ and F, respectively. Use the Fatou lemma to show that $F(x) \leq \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.

2.6 For a γ-detector, the times $\tau_1, \tau_2, \ldots$ that elapse between consecutive hits are independent random varibales which are exponentially distributed with $\mathbb{E}\tau_i = 1$. That is, their common density is $f(x) = e^{-x}\mathbf{1}_{[0,\infty)}(x)$ (where $\mathbf{1}$ stands for indicator function). (We measure time in seconds.)

Use the Cramer large deviation theorem to estimate the probability that we have to wait less than 500 seconds for the 1000-th hit.

2.7 **(homework)** Let $X_i, X_2, \ldots, X_n$ be independent random variables with Poisson distribution $X_i \sim Poi(\lambda)$. (That is, $\mathbb{P}(X_i = k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$.) Use the Cramer large deviation theorem to estimate the probability $\mathbb{P}(\sum_{k=1}^{n} X_k > 1000)$

(a) for $\lambda = 1$, $n = 500$,

(b) for $\lambda = 500$, $n = 1$.

2.8 *Change of measure in the proof of the Cramer theorem.* Let μ be a probability distribution on $\mathbb{R}$ and $Z(\lambda) := \int_{\mathbb{R}} e^{\lambda x}\, \mathrm{d}\mu(x)$ its moment generating function. Suppose that $Z(\lambda)$ is finite on the interval $(\underline{\lambda}, \overline{\lambda})$ with $\underline{\lambda} < 0 < \overline{\lambda}$. Let $\hat{I}(\lambda) := \log Z(\lambda)$, $y \in \mathbb{R}$ and suppose that $\lambda^* \in (\underline{\lambda}, \overline{\lambda})$ can be chosen such that $\hat{I}'(\lambda^*) = y$. Now let μ^* be the probability distribution on $\mathbb{R}$ which is absolutely continuous w.r.t. μ, and its density is $\frac{1}{Z(\lambda^*)}e^{\lambda^* x}$ – that is,

$$\mathrm{d}\mu^*(x) = \frac{1}{Z(\lambda^*)} e^{\lambda^* x}\, \mathrm{d}\mu(x).$$

(a) **(homework)** Show that the expectation of μ^* is exactly y – that is, $\int_{\mathbb{R}} x \, \mathrm{d}\mu^* = y$. (Don't worry much about exchanging differentiation and integrals.)

(b) Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with distribution μ, and let $X_1^*, X_2^*, \ldots, X_n^*$ be i.i.d random variables with distribution μ^*. Denote the distribution of $S_n := X_1 + X_2 + \cdots + X_n$ by μ_n and the distribution of $S_n^* := X_1^* + X_2^* + \cdots + X_n^*$ by μ_n^*. Show that

$$\mathrm{d}\mu_n^*(x) = \frac{1}{Z(\lambda^*)^n} e^{\lambda^* x} \, \mathrm{d}\mu_n(x).$$

(Hint: condsider the *joint distribution* of $(X_1^*, X_2^*, \ldots, X_n^*)$ (on $\mathbb{R}^n$). How is this related to the joint distribution of $(X_1, X_2, \ldots, X_n)$?)