Mathematical Statistical Physics – LMU München, summer semester 2012

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Homework sheet 11 – due on 06.07.2012 – and exercises for the class on 29.06.2012

11.1 **(homework)** *Metric structure on the Ising phase space and continuity.* Consider the infinite Ising phase space $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$ with the metric

$$d(\sigma, \omega) := \sum_{i=1}^{\infty} 2^{-i} \mathbf{1}_{\sigma_{k(i)} \neq \omega_{k(i)}}$$

where k is some fixed bijection from $\mathbb{N} = \{1, 2, \dots\}$ to $\mathbb{Z}^d$.

Call a function $f : \Omega \to \mathbb{R}$ *local*, if it only depends on finitely many elements of the configuration, i.e. there is a finite $\Lambda \subset \mathbb{Z}^d$ such that $f(\omega) = f(\sigma)$ whenever $\omega|_\Lambda = \sigma|_\Lambda$.

Call a function $f : \Omega \to \mathbb{R}$ *quasilocal*, if there is a sequence f_n of local functions such that $\|f_n - f\| \to 0$, where $\|.\|$ denotes the supremum norm.

Show that

$$f \text{ is continuous } \Leftrightarrow \forall \varepsilon > 0 \, \exists \Lambda \text{ finite } : \sup_{\sigma \in \Omega_\Lambda, \omega, \omega' \in \Omega_{\Lambda^c}} \|f(\sigma\omega) - f(\sigma\omega')\| < \varepsilon \Leftrightarrow f \text{ is quasilocal.}$$

11.2 **(homework)** *Consistency of the (grand) canonical measure with boundary condition for the Ising model.* Let $\gamma_\Lambda^{\beta,h}(A|\eta)$ denote the canonical measure of the set of configurations A under the boundary condition η, with box Λ for a nearest-neighbour Ising model. Show that whenever $\Lambda_1 \subset \Lambda_2$, we have

$$\gamma_{\Lambda_2}(A|\eta') = \int \gamma_{\Lambda_1}(A|\eta) \gamma_{\Lambda_2}(\,\mathrm{d}\eta|\eta').$$

11.3 *Symmetries of the Ising model.* Find all symmetries of the Ising model on $\mathbb{Z}^2$ with the simplest nearest neighbour interaction (without external filed)

$$J(\{i, \sigma_i\}, \{j, \sigma_j\}) = \begin{cases} -\sigma_i \sigma_j, & \text{if } |i - j| = 1 \\ 0, & \text{if not} \end{cases}$$

for every $i, j \in \mathbb{Z}^d$ and $\sigma_i, \sigma_j \in \{-1, 1\}$.

11.4 **(homework)** *Ground states of the Ising model.* Find all isolated and non-isolated ground states for the Ising model of the previous exercise in $d = 1$. Find all isolated ground states in $d = 2$. (Hint: show that there's nothing else than what we saw on the lecture.) In $d = 2$, find as many non-isolated ground states as you can. Have you found them all?

11.5 **(homework)** *Curie-Weiss model.* Consider the Ising-like model on $\Omega_N := \{-1, 1\}^N$ with the Hamiltonian

$$H_N(\sigma) := -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i.$$

(There are no boundary conditions.) Calculate the limiting thermodynamic pressure

$$p(\beta, h) := \lim_{N \to \infty} \frac{1}{\beta N} \log Z_N(\beta, h)$$

as explicitly as possible.

Study the continuity and analiticity of $p(\beta, h)$ — i.e. the existence of phase transitions. Find the critical temperature $\beta_c = \frac{1}{T_c}$.

Hint: we have shown in class that

$$\beta p(\beta, h) = \sup_{x \in (-1,1)} f_{\beta,h}(x)$$

where

$$f_{\beta,h}(x) = -\frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2} + \frac{\beta}{2} x^2 + \beta h x.$$

Draw the graph of $f_{\beta,h}(x)$ for different values of x.

Describe the behaviour of the magnetization $m = \frac{\partial p}{\partial h}$ and the susceptibility $\chi := \frac{1}{\beta} \frac{\partial m}{\partial h}$ near T_c – that is, calculate the "critical exponents" of the power-law behaviour. In particular, find the numbers b, γ, γ' and δ for which, around the critical point $(T, h) = (T_c, 0)$ we have

$$\begin{aligned}
m(T, 0+) &\sim |T - T_c|^{b} && \text{as} \quad T \nearrow T_c \\
\chi(T, 0+) &\sim |T - T_c|^{-\gamma} && \text{as} \quad T \searrow T_c \\
\chi(T, 0+) &\sim |T - T_c|^{-\gamma'} && \text{as} \quad T \nearrow T_c \\
|m(T_c, h)| &\sim |h|^{1/\delta} && \text{as} \quad h \to 0
\end{aligned}$$

What does "$\sim$" exactly mean here?

(Remark: the exponent b is usually denoted β, but now we better avoid confusion with $\beta = \frac{1}{T}$.)

Hint: The inverse functions are easy to write out and Taylor-expand (or differentiate). Using β instead of T is equally good, and the exponents will be the same (please check).