Mathematical Statistical Physics – LMU München, summer semester 2012

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Homework sheet 13 – due on 20.07.2012 – and exercises for the class on 13.07.2012

13.1 Prove a continuous time version (about endomorphism semigroups) of the von Neumann L^2 ergodic theorem using the discrete time version (about endomorphisms). *Hint: define* $F(x) := \int_0^1 f(S^s)\,\mathrm{d}s$.

13.2 **(homework)** Let $(M, \mathcal{F}, T, \mu)$ be an endomorphism and $n \in \mathbb{N}$.

(a) Show that if T^n is ergodic (w.r.t. μ), then T is also ergodic.

(b) Show that the converse is not true in general.

13.3 *Rotation of the circle.* Consider the phase space $S := \mathbb{R}/\mathbb{Z}$, which is a cirlce (or, if you like, a 1-dimensional torus, or the unit interval with periodic boundary conditions), with Lebesgue measure and the map $T : x \mapsto x + \alpha$ where $\alpha \in \mathbb{R}$.

(a) Show that T is an endomorphism.

(b) Show that T is ergodic iff α is irrational.

13.4 *Weil theorem.* Show the following: Let $S = \mathbb{R}/\mathbb{Z}$, $\alpha \in \mathbb{R}$ irrational and $I \subset S$ an interval. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_I(\{k\alpha\}) = \mathrm{Leb}(I).$$

Here $\{k\alpha\}$ denotes the fractional part of $k\alpha$, and could be written just as $k\alpha$, since we consider this as $k\alpha \in S$ anyway.

Hint: use the ergdicity from the previous exercise and approximate indicator functions by continuous ones.

13.5 **(homework)** (An exercise of Arnold.) Consider the first digits (in base 10) of the sequence of numbers $1, 2, 4, \dots, 2^n, \dots$. Does 7 occur? Does 8 occur? Which one occurs more often? *Hint:* $\log_{10} 2$ *is irrational. Look at the previous exercise.*

13.6 On $S = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure, consider the endomorphism group (flow) $S^t(x) := x + \alpha t$ where $\alpha \in \mathbb{R}$. Show that this is ergodic for every $\alpha \neq 0$. (Remark: actually this is even *uniquely ergodic*, meaning that this flow has only one invariant measure.)

13.7 *Perron-Frobenius operator*

(a) Let $(M, \mathcal{F}, T)$ be an endomorphism and assume that x is a random point in M distributed as some measure ν on M. What is the distribution of Tx?

(b) Let $T : [0, 1] \to [0, 1]$ be piecewise monotone and almost everywhere differentiable. Let x be a random point of $[0, 1]$ with probability density ρ (w.r.t. Lebesgue measure). Show that Tx also has a density and calcualte it.

(c) How does this work in higher dimensions?

13.8 **(homework)** *Gauss map.* Consider the map $T : (0, 1] \to (0, 1]$ defined as $Tx := \frac{1}{x}(mod 1)$. Show that the measure on $(0, 1]$ with density $\frac{const}{1+x}$ is invariant.

13.9 **(homework)** *The simplest possible convergence to equilibrium.* Consider the map $T : S \to S$, $Tx = 2x$. Let ν be a measure on S which is absolutely continuous w.r.t. Lebesgue measure, with a density φ which is Lipschitz continuous meaning

$$Lip(\varphi) := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|dist(x - y)|} < \infty.$$

Show that $\nu_n := \nu \circ T^{-n}$ converges to Lebesgue measure weakly.

Hint: What happens with $Lip(\varphi)$ under time evolution?

13.10 *Perron-Frobenius operator, invertible dynamics and entropy*

(a) Let $(M, \mathcal{F}, T, \mu)$ be an endomorphism and assume that x is a random point in M distributed as some measure ν on M, but possibly $\nu \neq \mu$. Assume however, that $\nu \ll \mu$ with density ρ. Show that $Tx \ll \mu$ as well, and calculate the density.

(b) $(M, \mathcal{F}, T, \mu)$ be an *auto*morphism and ν a measure on $(M, \mathcal{F})$ such that $\nu \ll \mu$. Define ν_n as $\nu_n := (\hat{T}^*)^n \nu = \nu \circ T^{in}$. Calculate the relative entropy $S_n = H(\nu_n|\mu)$.

(c) How come?