Mathematical Statistical Physics – LMU München, summer semester 2012

Hartmut Ruhl, Imre Péter Tóth

Homework sheet 11 – solutions

11.1 **(homework)** *Metric structure on the Ising phase space and continuity.* Consider the infinite Ising phase space $\Omega = \{-1,1\}^{\mathbb{Z}^d}$ with the metric

$$d(\sigma,\omega) := \sum_{i=1}^{\infty} 2^{-i} \mathbf{1}_{\sigma_{k(i)} \neq \omega_{k(i)}}$$

where k is some fixed bijection from $\mathbb{N} = \{1,2,\dots\}$ to $\mathbb{Z}^d$.

Call a function $f : \Omega \to \mathbb{R}$ *local*, if it only depends on finitely many elements of the configuration, i.e. there is a finite $\Lambda \subset \mathbb{Z}^d$ such that $f(\omega) = f(\sigma)$ whenever $\omega|_\Lambda = \sigma|_\Lambda$.

Call a function $f : \Omega \to \mathbb{R}$ *quasilocal*, if there is a sequence f_n of local functions such that $\|f_n - f\| \to 0$, where $\|.\|$ denotes the supremum norm.

Show that

$$f \text{ is continuous } \Leftrightarrow \forall \varepsilon > 0\, \exists \Lambda \text{ finite } : \sup_{\sigma\in\Omega_\Lambda, \omega,\omega'\in\Omega_{\Lambda^c}} |f(\sigma\omega) - f(\sigma\omega')| < \varepsilon \Leftrightarrow f \text{ is quasilocal.}$$

Solution: First of all, we know from the lecture that Ω is compact, so f is continuous iff it is equicontinuous, which is equivalent to

$$\forall \varepsilon > 0\, \exists n \in \mathbb{N} \text{ s.t. } d(\eta_1,\eta_2) < \frac{1}{2^n} \text{ implies } |f(\eta_1) - f(\eta_2)| < \varepsilon. \tag{1}$$

Let

$$\Lambda_n := \{k(1), k(2), \dots, k(n)\} \subset \mathbb{Z}^d \tag{2}$$

denote the set of points whose spins are counted with a weight at least $\frac{1}{2^n}$ in the definition of the metric. From the definition of the metric, $d(\eta_1,\eta_2) < \frac{1}{2^n}$ iff η_1 and η_2 coincide on Λ_n – let's denote that by $\eta_1|_{\Lambda_n} = \eta_2|_{\Lambda_n}$. [1]

- So to show the $\Rightarrow$ direction of the first equivalence, $\Lambda := \Lambda_n$ will do, where n is given by (1).
- To show the $\Leftarrow$ direction of the first equivalence, given a Λ there is surely an n for which $\Lambda \subset \Lambda_n$, and this n will do, since if $d(\eta_1,\eta_2) < \frac{1}{2^n}$ then $\eta_1|_{\Lambda_n} = \eta_2|_{\Lambda_n}$, which of course implies $\eta_1|_\Lambda = \eta_2|_\Lambda$, which by the assumption implies $|f(\eta_1) - f(\eta_2)| < \varepsilon$.

For the second equivalence: if f is quasilocal, then for every ε there is an f_n local such that $\|f - f_n\| < \frac{\varepsilon}{2}$. Assume (without loss of generality) that this local function f_n depends only on the spins in Λ_n defined in (2). Now

- to see the $\Leftarrow$ direction of the second equivalence, if $\eta_1|_{\Lambda_n} = \eta_2|_{\Lambda_n}$, then

  $$|f(\eta_1) - f(\eta_2)| \le |f(\eta_1) - f_n(\eta_1)| + |f_n(\eta_1) - f_n(\eta_2)| + |f_n(\eta_2) - f(\eta_2)| \le \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon,$$

  so $\Lambda = \Lambda_n$ will do.

[1] More precisely, $d(\eta_1,\eta_2) < \frac{1}{2^n}$ implies that $\eta_1|_{\Lambda_n} = \eta_2|_{\Lambda_n}$, and $\eta_1|_{\Lambda_n} = \eta_2|_{\Lambda_n}$ implies $d(\eta_1,\eta_2) \le \frac{1}{2^n}$, but this is not worth worrying about.

- to see the $\Rightarrow$ direction of the second equivalence, fix any $\omega \in \Omega$ and define f_n to be

$$f_n(\eta) := f(\eta|_{\Lambda_n}\omega|_{\Lambda_n^c}).$$

This f_n is clearly local (depends only the spins in Λ_n). For any ε the assumption gives a finite Λ, so for every n with $\Lambda \subset \Lambda_n$ we have

$$|f_n(\eta) - f(\eta)| = |f(\eta|_{\Lambda_n}\omega|_{\Lambda_n^c}) - f(\eta|_{\Lambda_n}\eta|_{\Lambda_n^c})| < \varepsilon.$$

11.2 **(homework)** *Consistency of the (grand) canonical measure with boundary condition for the Ising model.* Let $\gamma_\Lambda^{\beta,h}(A|\eta)$ denote the canonical measure of the set of configurations A under the boundary condition η, with box Λ for a nearest-neighbour Ising model. Show that whenever $\Lambda_1 \subset \Lambda_2$, we have

$$\gamma_{\Lambda_2}(A|\eta') = \int \gamma_{\Lambda_1}(A|\eta)\gamma_{\Lambda_2}(\,\mathrm{d}\eta|\eta').$$

Solution: As a reminder: for any given box Λ and boundary condition $\eta \in \Omega = \{-1,1\}^{\mathbb{Z}^d}$, the measure $\gamma_\Lambda^{\beta,h}(A|\eta)$ is a measure on the entire Ω, but it is concentrated on a finite set – namely the set of those configurations that coincide with η outside Λ. In particular, for every $\sigma \in \Omega$

$$\gamma_\Lambda^{\beta,h}(\{\sigma\}|\eta) := \begin{cases} \frac{1}{Z_\Lambda(\beta,h|\eta)}e^{-\beta H_\Lambda(\sigma|\eta)}, & \text{if } \sigma|_{\Lambda^c} = \eta|_{\Lambda^c}, \\ 0, & \text{if not,} \end{cases} \tag{3}$$

and these finitely many probabilities determine the measure $\gamma_\Lambda^{\beta,h}(.|\eta)$ completely. Here $H_\Lambda(\sigma|\eta)$ denotes the conditional Hamiltonian of the system in Λ, with boundary condition η, evaluated at σ. In the case of the nearest neighbour Ising model

$$H_\Lambda(\sigma|\eta) := -\frac{J}{2}\sum_{i,j\in\Lambda, i\sim j}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i - J\sum_{i\in\Lambda, j\in\Lambda^c, i\sim j}\sigma_i\eta_j, \tag{4}$$

where $i \sim j$ means "i and j are neighbours".

As a first goal, we patiently write out explicitly the "compatibility" equation we have to prove, using the definition (3). The equation contains an integral, but in our case this integral is just a finite sum, since we are integrating w.r.t. a measure that is concentrated on a finite set. Furthermore, since the measure on the left hand side is concentrated on a finite set, it is enough to check the equation for 1-element sets $A = \{\sigma\}$. That is, we have to show for every $\sigma \in \Omega$ that

$$\gamma_{\Lambda_2}(\{\sigma\}|\eta') = \sum_{\omega\in\Omega_{\Lambda_2}} \gamma_{\Lambda_1}(\{\sigma\}|\omega\eta'_{|\Lambda_2^c})\gamma_{\Lambda_2}(\{\omega\eta'_{|\Lambda_2^c}\}|\eta').$$

If $\sigma_{|\Lambda_2^c} \neq \eta'_{|\Lambda_2^c}$, then the left hand side is zero, but $\Lambda_1 \subset \Lambda_2$ implies that $\sigma_{|\Lambda_1^c} \neq \eta'_{|\Lambda_1^c}$ as well, so the right hand side is also zero, and the equation holds. So from now on we assume that $\sigma_{|\Lambda_2^c} = \eta'_{|\Lambda_2^c}$. With that assumed, there are still a lot of zero terms on the right hand side: namely those ω-s that don't coincide with σ on $\Lambda_2 \setminus \Lambda_1$, all contribute zero. To reflect this fact we rewrite the sum to run only over ω-s of the form $\omega = \xi\sigma_{|\Lambda_2\setminus\Lambda_1}$ with $\xi \in \omega_{\Lambda_1}$. This way $\omega\eta'_{|\Lambda_2^c}$ becomes $\xi\sigma_{|\Lambda_2\setminus\Lambda_1}\eta'_{|\Lambda_2^c}$, but since we already fixed $\sigma_{|\Lambda_2^c} = \eta'_{|\Lambda_2^c}$, we can write this as $\omega\eta'_{|\Lambda_2^c} = \xi\sigma_{|\Lambda_1^c}$. So we get that we have to show

$$\gamma_{\Lambda_2}(\{\sigma\}|\eta') = \sum_{\xi\in\Omega_{\Lambda_1}} \gamma_{\Lambda_1}(\{\sigma\}|\xi\sigma_{|\Lambda_1^c})\gamma_{\Lambda_2}(\{\xi\sigma_{|\Lambda_1^c}\}|\eta').$$

Now all factors in all terms are nonzero and we can write this using the definition (3) as

$$\frac{1}{Z_{\Lambda_2}(\beta,h|\eta')}e^{-\beta H_{\Lambda_2}(\sigma|\eta')} = \sum_{\xi\in\Omega_{\Lambda_1}} \frac{1}{Z_{\Lambda_1}(\beta,h|\xi\sigma_{|\Lambda_1^c})}e^{-\beta H_{\Lambda_1}(\sigma|\xi\sigma_{|\Lambda_1^c})}\frac{1}{Z_{\Lambda_2}(\beta,h|\eta')}e^{-\beta H_{\Lambda_2}(\xi\sigma_{|\Lambda_1^c}|\eta')}.$$

The constant divisor $Z_{\Lambda_2}(\beta,h|\eta')$ cancels out and we are left with

$$e^{-\beta H_{\Lambda_2}(\sigma|\eta')} = \sum_{\xi\in\Omega_{\Lambda_1}} \frac{1}{Z_{\Lambda_1}(\beta,h|\xi\sigma_{|\Lambda_1^c})}e^{-\beta H_{\Lambda_1}(\sigma|\xi\sigma_{|\Lambda_1^c})}e^{-\beta H_{\Lambda_2}(\xi\sigma_{|\Lambda_1^c}|\eta')}.$$

With that we accomplished the first goal, and it's time to look at the definition (4) of H. We first observe that $H_{\Lambda_2}(.|\eta')$ only depends on the values of η' outside Λ_2, where we fixed it to be equal to σ, so $H_{\Lambda_2}(.|\eta') = H_{\Lambda_2}(.|\sigma)$. Similarly, $H_{\Lambda_1}(.|\xi\sigma_{|\Lambda_1^c}) = H_{\Lambda_1}(.|\sigma)$ and consequently $Z_{\Lambda_1}(\beta,h|\xi\sigma_{|\Lambda_1^c}) = Z_{\Lambda_1}(\beta,h|\sigma)$, so the equation can be written in the little friendlier form

$$e^{-\beta H_{\Lambda_2}(\sigma|\sigma)} = \sum_{\xi\in\Omega_{\Lambda_1}} \frac{1}{Z_{\Lambda_1}(\beta,h|\sigma)}e^{-\beta H_{\Lambda_1}(\sigma|\sigma)}e^{-\beta H_{\Lambda_2}(\xi\sigma_{|\Lambda_1^c}|\sigma)},$$

which in particluar does not contain η'. To match the two sides, it is a natural idea to use the fact that Z is a normalizing factor in the form

$$\sum_{\xi\in\Omega_{\Lambda_1}} \frac{1}{Z_{\Lambda_1}(\beta,h|\sigma)}e^{-\beta H_{\Lambda_1}(\xi*|\sigma)} = 1,$$

where $*$ can denote any element of $\Omega_{\Lambda_1^c}$, $H_{\Lambda_1}(\xi*|\sigma)$ does not depend on it. So our equation becomes

$$\sum_{\xi\in\Omega_{\Lambda_1}} e^{-\beta H_{\Lambda_1}(\xi*|\sigma)}e^{-\beta H_{\Lambda_2}(\sigma|\sigma)} = \sum_{\xi\in\Omega_{\Lambda_1}} e^{-\beta H_{\Lambda_1}(\sigma|\sigma)}e^{-\beta H_{\Lambda_2}(\xi\sigma_{|\Lambda_1^c}|\sigma)}.$$

If we are lucky, the two sums are equal term by term – and that's exactly what happens: it is enough to check that for every $\xi\in\Omega_{\Lambda_1}$ and every $\sigma\in\Omega$

$$H_{\Lambda_1}(\xi*|\sigma) + H_{\Lambda_2}(\sigma|\sigma) = H_{\Lambda_1}(\sigma|\sigma) + H_{\Lambda_2}(\xi\sigma_{|\Lambda_1^c}|\sigma)$$

holds. This can be seen directly from the definition (4) by matching the terms of the sums on the two sides.

11.3 *Symmetries of the Ising model.* Find all symmetries of the Ising model on $\mathbb{Z}^2$ with the simplest nearest neighbour interaction (without external field)

$$J(\{i,\sigma_i\},\{j,\sigma_j\}) = \begin{cases} -\sigma_i\sigma_j, & \text{if } |i-j|=1 \\ 0, & \text{if not} \end{cases}$$

for every $i,j\in\mathbb{Z}^d$ and $\sigma_i,\sigma_j\in\{-1,1\}$.

11.4 **(homework)** *Ground states of the Ising model.* Find all isolated and non-isolated ground states for the Ising model of the previous exercise in $d=1$. Find all isolated ground states in $d=2$. (Hint: show that there's nothing else than what we saw on the lecture.) In $d=2$, find as many non-isolated ground states as you can. Have you found them all?

Solution:

(a) For $d = 1$ this was done in class: a ground state can contain no island, so in $d = 1$ there can be at most one boundary between a region of +-es and a region of −-es.

- The two configurations with all spins equal are isolated ground states, since changing them in any finite box creates at least one island (two boundaries) and thus increases the energy.
- The configurations with exactly one boundary ($\sigma(k) = \mathbf{1}_{\{k \le n\}} - \mathbf{1}_{\{k > n\}}$ or the negative of this) are also ground states, since if we change them in a finite box, the boundary can't disappear. However, they are not isolated ground states, since it's possible to change them so that the boundary is only shifted, so the energy doesn't change.

(b) For $d = 2$ the picture is much more complex. Here, instead of "boundaries", it's better to talk about "contours" separating regions of +-es and −-es, which contribute to the energy.

- (partially done in class) Also now, the two configurations with all spins equal are isolated ground states, since changing them in any finite box creates at least one island surrounded by a contour, and thus increases the energy. However, there are also others: any configuration with a single *straight* contour is also an isolated ground state, since changing it in a finite box either creates a new contour or increases the length of this contour, and thus increases the energy. There are no more isolated ground states, since
  - A non-straight contour always allows for a local change which doesn't increase the energy.
  - Two straight contours cannot be perpendicular and intersect, because then they aren't really contours – e.g. in the configuration

$$\sigma(i,j) := \begin{cases} +1, & \text{if } i,j \ge 0 \text{ or } i,j < 0 \\ -1, & \text{if not} \end{cases} \tag{5}$$

    the contours (separating +-es from −-es) are actually two broken lines, and this is not an isolated ground state.
  - If there are two parallel straight contours, then changing all spins between the two lines in a big enough box will decrease the energy.
- Concerning non-isolated ground states, clearly no contour can be a closed curve, since then the entire island surrounded by it could be flipped and the energy would decrease. So countours should be infinite curves. It is easy to see that only *monotone* curves are allowed in the sense that any finite piece can be drawn (as a continuous curve) either by using only "right" and "up" steps, or using only "right" and "down" steps. [2] Indeed, if a contour is not monotone, it can be made shorter with a local change, causing the energy to decrease.
  - It is clear that if only 1 monotone contour is present (separating a "half-plane" of +-es from a "half-plane" of −-es), then the configuration is a ground state, and non-isolated unless the contour happens to be a (horizontal or vertical) straight line.
  - One could guess that there are no more ground states, but this is not the case: it is indeed possible – with strong restrictions – to have two monotone contours

[2]This is the same as saying that the set of lattice points on one side of the contour should be either "monotone" in the sense that if a point P is in it, then every point which is above or to the right of P is also in it, or the rotation of such a monotone set.

present in a (non-isolated) ground state. I will not give the precise condition here (it's not very hard), but mention the following examples (please draw):

  * $\sigma(i,j) := \begin{cases} +1, & \text{if } i,j > 10 \text{ or } i,j < -10 \\ -1, & \text{if not} \end{cases}$
  * $\sigma(i,j) := \begin{cases} +1, & \text{if } i \geq 100,\ j \geq 0 \text{ or } i < -100, j < 10 \\ -1, & \text{if not} \end{cases}$
  * the configuration in (5).

- It's not possible to have more than two contours peresent (not hard to see).

11.5 **(homework)** *Curie-Weiss model.* Consider the Ising-like model on $\Omega_N := \{-1,1\}^N$ with the Hamiltonian

$$H_N(\sigma) := -\frac{1}{2N}\sum_{i,j=1}^{N} \sigma_i\sigma_j - h\sum_{i=1}^{N} \sigma_i.$$

(There are no boundary conditions.) Calculate the limiting thermodynamic pressure

$$p(\beta,h) := \lim_{N\to\infty} \frac{1}{\beta N} \log Z_N(\beta,h)$$

as explicitly as possible.

Study the continuity and analiticity of $p(\beta,h)$ — i.e. the existence of phase transitions. Find the critical temperature $\beta_c = \frac{1}{T_c}$.

Hint: we have shown in class that

$$\beta p(\beta,h) = \sup_{x\in(-1,1)} f_{\beta,h}(x)$$

where

$$f_{\beta,h}(x) = -\frac{1+x}{2}\log\frac{1+x}{2} - \frac{1-x}{2}\log\frac{1-x}{2} + \frac{\beta}{2}x^2 + \beta h x.$$

Draw the graph of $f_{\beta,h}(x)$ for different values of x.

Describe the behaviour of the magnetization $m = \frac{\partial p}{\partial h}$ and the susceptibility $\chi := \frac{1}{\beta}\frac{\partial m}{\partial h}$ near T_c – that is, calculate the "critical exponents" of the power-law behaviour. In particular, find the numbers b, γ, γ' and δ for which, around the critical point $(T,h) = (T_c,0)$ we have

$$\begin{aligned} m(T,0+) &\sim |T-T_c|^{b} && \text{as} \quad T \nearrow T_c \\ \chi(T,0+) &\sim |T-T_c|^{-\gamma} && \text{as} \quad T \searrow T_c \\ \chi(T,0+) &\sim |T-T_c|^{-\gamma'} && \text{as} \quad T \nearrow T_c \\ |m(T_c,h)| &\sim |h|^{1/\delta} && \text{as} \quad h \to 0 \end{aligned}$$

What does "$\sim$" exactly mean here?

(Remark: the exponent b is usually denoted β, but now we better avoid confusion with $\beta = \frac{1}{T}$.)

Hint: The inverse functions are easy to write out and Taylor-expand (or differentiate). Using β instead of T is equally good, and the exponents will be the same (please check).

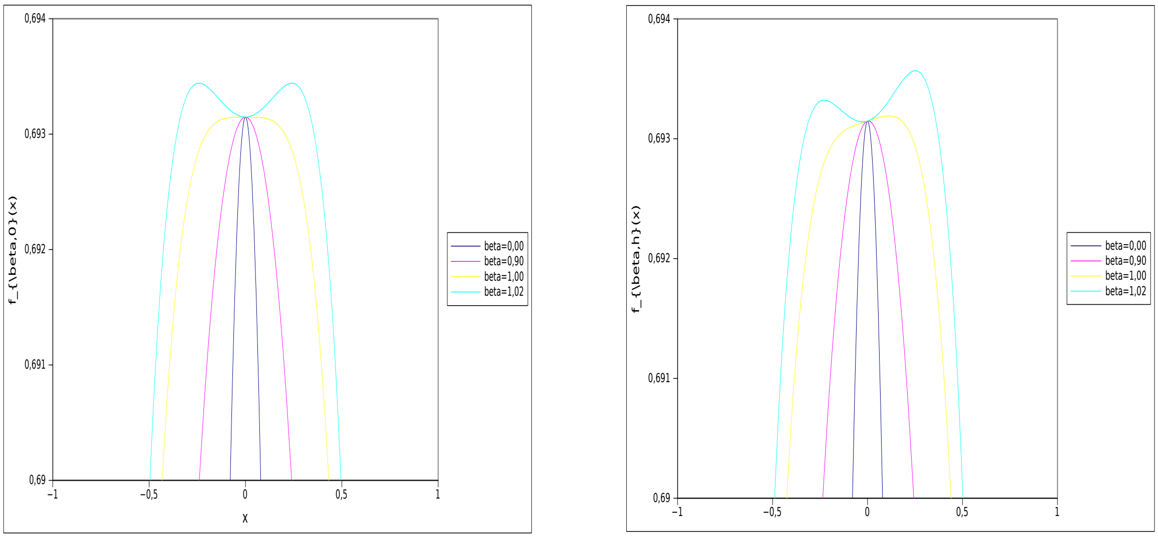
Solution: First draw the graph of $f_{\beta,h}(x)$ for interesting values of β and h. As a start, set $h = 0$, so $f_{\beta,0}(x)$ is an even function and

$$\begin{aligned} f'_{\beta,0}(x) &= -\frac{1}{2}\log(1+x) + \frac{1}{2}\log(1-x) + \beta x, \\ f''_{\beta,0}(x) &= \frac{-1}{1-x^2} + \beta. \end{aligned}$$

Thus the second derivative goes to $-\infty$ as $x \to \pm 1$ and has its maximum at $x = 0$. As a result, $f_{\beta,0}(x)$ is concave near ± 1 (with derivatives going to $\mp\infty$), and

- for $\beta \leq 1$ it is concave everywhere, thus having a single maximum at 0,
- This single maximum is nondegenerate if $\beta < 1$ (with nonzero second derivative), while degenerate for $\beta = 1$.
- for $\beta > 1$ it is convex near 0, thus having a minimum at 0 and two (equal) local (and global) maximua at $\pm m_*(\beta, 0) \neq 0$.

See Figure 1(a). The critical temperature is clearly $\beta_c = 1$.



(a) $h = 0$

(b) $h = 0.0005$

Figure 1: $f_{\beta,h}(x)$ for different values of β and h

If h is nonzero, a linear term is added, tilting the graph. See Figure 1(b).

- If $\beta \leq 1$, this only moves the single maximum, leaving the function concave.
- If $\beta > 1$ and $|h|$ is small enough, there will still be two local maxima near $\pm m_*(\beta, 0)$, but they will no longer be equal. The value of the maximum will be the bigger local maximum, which is nondegenerate (meaning $f'' < 0$).
- If $\beta > 1$, as $|h|$ gets larger, the smaller local maximum gets degenerate and disappears, but the bigger (later only) one stays nondegenerate.

Let's denote the place of the global maximum by $m_*(\beta, h)$, which is of course also a stationary point, so it satisfies

$$0 = f'_{\beta,h}(m_*(\beta, h)) = f'_{\beta,0}(m_*(\beta, h)) + \beta h. \tag{6}$$

The global maximum $m_*(\beta,h)$ is well defined unless $\beta > 1$ and $h = 0$. In this case

$$\beta p(\beta,h) = f_{\beta,h}(m_*(\beta,h)),$$

so the magnetization is

$$\begin{aligned} m(\beta,h) = \frac{\partial}{\partial h} p(\beta,h) &= \frac{1}{\beta}\left(\frac{\partial}{\partial h} f_{\beta,h}(m_*(\beta,h)) + f'_{\beta,h}(m_*(\beta,h))\frac{\partial}{\partial h} m_*(\beta,h)\right) = \\ &= m_*(\beta,h) + \frac{1}{\beta}\cdot 0 \cdot \frac{\partial}{\partial h} m_*(\beta,h) = m_*(\beta,h) \end{aligned} \tag{7}$$

if only $\frac{\partial}{\partial h} m_*(\beta,h)$ exists. To see that it exists for every $h \neq 0$, we calculate the implicite derivative from the equation (6) as

$$0 = f''_{\beta,0}(m_*(\beta,h))\frac{\partial}{\partial h} m_*(\beta,h) + \beta. \tag{8}$$

So $\frac{\partial}{\partial h} m_*(\beta,h)$ exists and is finite whenever $f''_{\beta,0}(m_*(\beta,h)) \neq 0$, which holds for all $h \neq 0$ (and also for $\beta < 1, h = 0$), because then the global maximum is nondegenerate.

At $\beta > 1, h = 0$ the argument in (7) still works for one-sided derivatives, so

$$\begin{aligned} \lim_{h \searrow 0} \frac{p(\beta,h) - p(\beta,0)}{h} &= m_*(\beta,0) \\ \lim_{h \nearrow 0} \frac{p(\beta,h) - p(\beta,0)}{h} &= -m_*(\beta,0) \end{aligned} \tag{9}$$

and $p(\beta,h)$ is not differentiable w.r.t. h at 0.

From the above we can also read out the analiticity of p:

- p is analytic in h whenever there is a single nondegenerate maximum of f – that is, everywhere except when $\beta \geq 1$ and $h = 0$.
- The β-dependence of p is easier and less interesting: with h fixed, $p(\beta,h)$ depends analytically on β everywhere except for $(h = 0, \beta = 1)$.

To calculate the critical exponents, first notice that (9) says exactly that the magnetization at $h = 0+$ at low temperature is exactly $m := m(\beta, 0+) = m_*(\beta,0)$. Writng out (6) explicitely, this is given by

$$0 = f'_{\beta,0}(m) = -\frac{1}{2}\log(1+m) + \frac{1}{2}\log(1-m) + \beta m,$$

so the inverse function $\beta(m)$ is easy to calculate and Taylor-expand:

$$\begin{aligned} \beta(m) &= \frac{\log(1+m) - \log(1-m)}{2m} = \\ &= \frac{\left(m - \frac{m^2}{2} + \frac{m^3}{3} + o(m^3)\right) - \left(-m - \frac{m^2}{2} - \frac{m^3}{3} + o(m^3)\right)}{2m} = \\ &= 1 + \frac{2m^2}{3} + o(m^2). \end{aligned} \tag{10}$$

So

$$|T - T_c| = \left|\frac{1}{\beta} - \frac{1}{\beta_c}\right| = \frac{|\beta - \beta_c|}{|\beta\beta_c|} \sim |\beta - \beta_c| \sim m^2,$$

where $\sim$ means exactly that the ratio of the two sides converges to a nonzero constant (in this case as $\beta \searrow \beta_c$). After all,

$$m \sim |T - T_c|^{1/2} \quad \text{and so} \quad b = \frac{1}{2}.$$

For the critical exponents γ and γ' one needs the susceptibility $\chi := \frac{1}{\beta}\frac{\partial m}{\partial h}$, but since $m = m_*$ we have already calculated this in (8), which can be written as

$$0 = f''_{\beta,0}(m)\beta\chi + \beta,$$

so

$$\frac{1}{\chi} = -f''_{\beta,0}(m) = \frac{1}{1-m^2} - \beta.$$

- For $\beta < 1$ ($T > T_c$) we have $m(\beta, 0) = 0$, so $\frac{1}{\chi(\beta,0)} = 1 - \beta = |\beta - \beta_c| \sim |T - T_c|$ and $\gamma = 1$.
- For $\beta > 1$ ($T < T_c$) we know from (10) that $m(\beta, 0+) = \frac{3}{2}(\beta - 1) + o(\beta - 1)$, so

$$\begin{aligned}\frac{1}{\chi(\beta,0+)} &= \frac{1}{1 - \frac{3}{2}(\beta-1) + o(\beta-1)} - \beta = \\ &= 1 + \frac{3}{2}(\beta - 1) + o(\beta - 1) - \beta = \\ &= \frac{1}{2}(\beta - 1) + o(\beta - 1) \sim |\beta - \beta_c| \sim |T - T_c|,\end{aligned}$$

  so $\gamma' = 1$ as well.

To get δ, we again write out (6) explicitely, this time with $\beta = 1$:

$$\begin{aligned}h(m) &= h(m, \beta = 1) = -f'_{\beta,0}(m) = \frac{1}{2}(\log(1+m) - \log(1-m)) - m = \\ &= \frac{m - \frac{m^2}{2} + \frac{m^3}{3} + o(m^3) - (-m - \frac{m^2}{2} - \frac{m^3}{3} + o(m^3))}{2} - m = \\ &= \frac{1}{3}m^3 + o(m^3) \sim m^3,\end{aligned}$$

so $m(\beta_c, h) \sim h^{\frac{1}{3}}$ and $\delta = 3$.