Mathematical Statistical Physics – LMU München, summer semester 2012

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Homework sheet 13 – solutions

13.1 Prove a continuous time version (about endomorphism semigroups) of the von Neumann L^2 ergodic theorem using the discrete time version (about endomorphisms). *Hint: define* $F(x) := \int_0^1 f(S^s)\,\mathrm{d}s$.

13.2 **(homework)** Let $(M, \mathcal{F}, T, \mu)$ be an endomorphism and $n \in \mathbb{N}$.

(a) Show that if T^n is ergodic (w.r.t. μ), then T is also ergodic.

(b) Show that the converse is not true in general.

Solution:

(a) Suppose that T^n is egrodic. Then for any T-invariant functioon f (meaning $f = f \circ T$ μ-almost surely) we have

$$f \circ T^{k+1} = (f \circ T) \circ T^k = f \circ T^k \ \mu\text{-almost surely},$$

so by induction $f = f \circ T^n$ μ-almost surely, so the ergodicity of T^n implies that f is constant μ-almost surely. We have shown that T is ergodic.

(b) Counterexample: let $M = \{a, b\}$ be a 2-element set, μ be uniform on M and let T exchange the two elements: $T(a) := b$, $T(b) := a$. Then T is clearly ergodic, since for every invariant function f, $f(a) = f(b)$ has to hold. On the other hand, $T^2 = Id$ is clearly not ergodic, since every function $f : M \to \mathbb{R}$ is invariant.

13.3 *Rotation of the circle.* Consider the phase space $S := \mathbb{R}/\mathbb{Z}$, which is a cirlce (or, if you like, a 1-dimensional torus, or the unit interval with periodic boundary conditions), with Lebesgue measure and the map $T : x \mapsto x + \alpha$ where $\alpha \in \mathbb{R}$.

(a) Show that T is an endomorphism.

(b) Show that T is ergodic iff α is irrational.

13.4 *Weil theorem.* Show the following: Let $S = \mathbb{R}/\mathbb{Z}$, $\alpha \in \mathbb{R}$ irrational and $I \subset S$ an interval. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_I(\{k\alpha\}) = \mathrm{Leb}(I).$$

Here $\{k\alpha\}$ denotes the fractional part of $k\alpha$, and could be written just as $k\alpha$, since we consider this as $k\alpha \in S$ anyway.

Hint: use the ergdicity from the previous exercise and approximate indicator functions by continuous ones.

13.5 **(homework)** (An exercise of Arnold.) Consider the first digits (in base 10) of the sequence of numbers $1, 2, 4, \ldots, 2^n, \ldots$. Does 7 occur? Does 8 occur? Which one occurs more often? *Hint:* $\log_{10} 2$ *is irrational. Look at the previous exercise.*

Solution: An easy calculation shows that the first digit of 2^n is $k \in \{1, 2, \ldots, 9\}$ if and only if $\{n \log_{10} 2\} \in I_k := [\log_{10} k, \log_{10}(k+1))$ where $\{.\}$ denotes fraction part. So the Weil theorem implies that the density of such n is $Leb(I_k) = \log_{10}(1 + \frac{1}{k})$. So every k, including 7 and 8 occurs, and 7 occurs more often that 8.

13.6 On $S = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure, consider the endomorphism group (flow) $S^t(x) := x + \alpha t$ where $\alpha \in \mathbb{R}$. Show that this is ergodic for every $\alpha \neq 0$. (Remark: actually this is even *uniquely ergodic*, meaning that this flow has only one invariant measure.)

13.7 *Perron-Frobenius operator*

(a) Let $(M, \mathcal{F}, T)$ be an endomorphism and assume that x is a random point in M distributed as some measure ν on M. What is the distribution of Tx?

(b) Let $T : [0,1] \to [0,1]$ be piecewise monotone and almost everywhere differentiable. Let x be a random point of $[0,1]$ with probability density ρ (w.r.t. Lebesgue measure). Show that Tx also has a density and calcualte it.

(c) How does this work in higher dimensions?

13.8 **(homework)** *Gauss map.* Consider the map $T : (0,1] \to (0,1]$ defined as $Tx := \frac{1}{x}(mod 1)$. Show that the measure on $(0,1]$ with density $\frac{const}{1+x}$ is invariant.

Solution: Set $I_n := (\frac{1}{n+1}, \frac{1}{n}]$ for $n = 1, 2, \dots$. So $Tx = \frac{1}{x} - n$ on I_n. and $|T'(x)| = \frac{1}{x^2}$ (almost everywhere). We apply the Perron-Frobenius operator to the density $\varphi(c) = \frac{c}{1+x}$ and get

$$\begin{aligned}(\mathcal{P}\varphi)(y) &:= \sum_{x:Tx=y} \frac{\varphi(x)}{|T'(x)|} = \sum_{n=1}^{\infty} \frac{\varphi(\frac{1}{y+n})}{\frac{1}{(y+n)^2}} = \sum_{n=1}^{\infty} \frac{c}{(y+n+1)(y+n)} = \\ &= \sum_{n=1}^{\infty} [\frac{c}{y+n} - \frac{c}{y+n+1}] = \frac{c}{y+1} = \varphi(y).\end{aligned}$$

In the last step we used the fact that the sum is telescopic:

$$\sum_{n=1}^{\infty} [\frac{c}{y+n} - \frac{c}{y+n+1}] = \lim_{N\to\infty} \sum_{n=1}^{N} [\frac{c}{y+n} - \frac{c}{y+n+1}] = \lim_{N\to\infty} [\frac{c}{y+1} - \frac{c}{y+n+1}] = \frac{c}{y+1}.$$

We have checked that $\mathcal{P}\varphi = \varphi$, so the mesure is invariant.

Remark: It is false to claim that $\sum_{n=1}^{\infty} [\frac{c}{y+n} - \frac{c}{y+n+1}] = \sum_{n=1}^{\infty} \frac{c}{y+n} - \sum_{n=1}^{\infty} \frac{c}{y+n+1}$, since both sums on the r.h.s. are divergent.

13.9 **(homework)** *The simplest possible convergence to equilibrium.* Consider the map $T : S \to S$, $Tx = 2x$. Let ν be a measure on S which is absolutely continuous w.r.t. Lebesgue measure, with a density φ which is Lipschitz continuous meaning

$$Lip(\varphi) := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|dist(x-y)|} < \infty.$$

Show that $\nu_n := \nu \circ T^{-n}$ converges to Lebesgue measure weakly.

Hint: What happens with $Lip(\varphi)$ under time evolution?

Solution: Let $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1)$ so $Tx = 2x$ on I_0 and $Tx = 2x - 1$ on I_1 and $|T'(x)| = 2$ almost everywhere. Appliing thePerron-Frobenius operator to a density φ gives

$$(\mathcal{P}\varphi)(y) := \sum_{x:Tx=y} \frac{\varphi(x)}{|T'(x)|} = \frac{\varphi(\frac{y}{2}) + \varphi(\frac{y+1}{2})}{2}.$$

We can estiame the Lipschitz constant of this $\mathcal{P}\varphi$ as

$$\begin{aligned}
|(\mathcal{P}\varphi)(y_1) - (\mathcal{P}\varphi)(y_2)| &= \frac{1}{2}\left|\varphi(\frac{y_1}{2}) + \varphi(\frac{y_1+1}{2}) - \varphi(\frac{y_2}{2}) - \varphi(\frac{y_2+1}{2})\right| = \\
&= \frac{1}{2}\left|\varphi(\frac{y_1}{2}) - \varphi(\frac{y_2}{2}) + \varphi(\frac{y_1+1}{2}) - \varphi(\frac{y_2+1}{2})\right| \leq \\
&\leq \frac{1}{2}\left|\varphi(\frac{y_1}{2}) - \varphi(\frac{y_2}{2})\right| + \frac{1}{2}\left|\varphi(\frac{y_1+1}{2}) - \varphi(\frac{y_2+1}{2})\right| \leq \\
&\leq \frac{1}{2}\left|\frac{y_1}{2} - \frac{y_1+1}{2}\right| Lip(\varphi) + \frac{1}{2}\left|\frac{y_2}{2} - \frac{y_2+1}{2}\right| Lip(\varphi) = \\
&= \frac{1}{2}\cdot 2 \cdot \frac{|y_1-y_2|}{2} Lip(\varphi),
\end{aligned}$$

which gives

$$Lip(\mathcal{P}\varphi) \leq \frac{1}{2} Lip(\varphi),$$

so $Lip(\mathcal{P}^n\varphi) \to 0$ as $n \to \infty$, which immediately implies $\sup \mathcal{P}^n\varphi - \inf \mathcal{P}^n\varphi \to 0$. On the other hand $\mathcal{P}^n\varphi$ is a probability density on $(0,1)$ which has to integrate to 1, which implies that $\inf \mathcal{P}^n\varphi \leq 1 \leq \sup \mathcal{P}^n\varphi$. Now these together imply that $\sup \mathcal{P}^n\varphi \to 1$ and $\inf \mathcal{P}^n\varphi \to 1$ as well, and $\mathcal{P}^n\varphi \to 1$ everywhere (uniformly). We know from Homework 2.5 that this implies weak convergence of the measures ν_n to Lebesgue measure.

13.10 *Perron-Frobenius operator, invertible dynamics and entropy*

(a) Let $(M, \mathcal{F}, T, \mu)$ be an endomorphism and assume that x is a random point in M distributed as some measure ν on M, but possibly $\nu \neq \mu$. Assume however, that $\nu \ll \mu$ with density ρ. Show that $Tx \ll \mu$ as well, and calculate the density.

(b) $(M, \mathcal{F}, T, \mu)$ be an *auto*morphism and ν a measure on $(M, \mathcal{F})$ such that $\nu \ll \mu$. Define ν_n as $\nu_n := (\hat{T}^*)^n\nu = \nu \circ T^{in}$. Calculate the relative entropy $S_n = H(\nu_n|\mu)$.

(c) How come?