Some basic conventions

Billiards

A billiard is a dynamical system describing the motion of a point particle in a connected, compact domain $Q \subset \mathbb{T}^d$. The boundary of the domain is assumed to be piecewise C^3-smooth. Inside Q the motion is uniform while reflection at the boundary ∂Q is elastic. As the absolute value of the velocity is a first integral of motion, the phase space of the billiard flow is fixed as $M = Q \times S^{d-1}$ – in other words, every phase point x is of the form $x = (q, v)$ with $q \in Q$ and $v \in \mathbb{R}^d$, $|v| = 1$. The Liouville probability measure μ on M is essentially the product of the Lebesgue measures, i.e. $d\mu = \text{const.}\, dqdv$. **The resulting dynamical system $(M, S^{\mathbb{R}}, \mu)$ is the (toric)** *billiard flow.*

The boundary ∂Q defines a natural cross-section, the **Poincaré section** for the billiard flow. Namely consider

$$\mathcal{M} = \partial M = \{(q, v) \mid q \in \partial Q, \langle v, n(q)\rangle \geq 0\},$$

the **Poincaré phase space** (This set actually has a natural bundle structure.) The Poincaré section map T, also called the *billiard map* of *Poincaré map* (denoted by $(\mathcal{M}, T, \mu_1)$) is defined as the first return map on $\mathcal{M}$. The invariant measure for the map is denoted by μ_1, and we have $d\mu_1 = \text{const.}\, |\langle v, n(q)\rangle|\, dqdv$. Recall the usual notation: for $(q, v) \in M$ one denotes $\pi(q, v) = q$, the natural projection.

Semi-dispersing and Dispersing billiards

Let $n(q)$ denote the unit normal vector of a smooth component of the boundary ∂Q at the point q, directed inwards Q. A billiard is called *semi-dispersing* (*dispersing*) if for every $q \in \partial Q$ the second fundamental form $K(q)$ of the boundary component is non-negative (positive, resp.).

The Lorentz process

The *Lorentz process* is the natural $\mathbb{Z}^d$ cover of a toric billiard. More precisely: consider $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ the factorisation by $\mathbb{Z}^d$. Its fundamental domain D is a d-dimensional cube (semi-open, semi-closed) in $\mathbb{R}^d$, so $\mathbb{R}^d = \cup_{z \in \mathbb{Z}^d}(D + z)$, where $D + z$ is the translated fundamental domain. By denoting $\tilde{Q} = \Pi^{-1}Q$, $\tilde{M} = \tilde{Q} \times S^{d-1}$, etc., the Lorentz dynamics is $(\tilde{M}, \tilde{S}^{\mathbb{R}}, \tilde{\mu})$ and its Poincaré section map is $(\partial\tilde{M}, \tilde{T}, \tilde{\mu}_1)$. (The Lorentz dynamics is actually a billiard whose invariant measure is infinite.)

By dropping the natural periodicity property of the scatterers of the 'periodic' Lorentz process one obtains the model of *the Lorentz process, in general.*

The Lorentz gas

The Lorentz gas is the motion of a countable number of non-interacting Lorentz particles. The natural invariant measure is the Gibbs-measure, which in this case is a Poissonian measure with density

$$\rho \ \exp(-v^2/2) \ dx \ dv.$$