Generalized Hölder continuity and oscillation functions

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Abstract

We study a notion of generalized Hölder continuity for functions on $\mathbb{R}^d$. We show that for any bounded function f of bounded support and any $r > 0$, the r-oscillation of f defined as $osc_r f(x) := \sup_{B_r(x)} f - \inf_{B_r(x)} f$ is automatically generalized Hölder continuous, and we give an estimate for the appropriate (semi)norm. This is motivated by applications in the theory of dynamical systems.

1 Introduction

Let $f : X \to \mathbb{R}$, where $(X, dist)$ is some metric space. Let $0 < \alpha \in \mathbb{R}$ and $0 \leq C < \infty$. The function f is said to be Hölder continuous with exponent α and Hölder constant C if for any $x, y \in X$

$$|f(x) - f(y)| \leq C dist(x, y)^\alpha. \tag{1.1}$$

We now consider $X := \mathbb{R}^d$ with the natural Euclidean metric. Following Keller [4], Saussol [5] and Chernov [2], we generalise the above notion so that (1.1) need not hold for every pair (x, y), only "on average" w.r.t Lebesgue measure. This is motivated by applications in the theory of dynamical systems: in the above quantitative studies of mixing (and also in others), such a generalized Hölder continuity turns out to be the correct notion of regularity, which we need to assume about observables.

For the sake of this paper, we will use $B_r(x)$ to denote the open ball of radius r centred at $x \in \mathbb{R}^d$:

$$B_r(x) := \{y \in \mathbb{R}^d \, : \, |y - x| < r\}.$$

Let $D \subset \mathbb{R}^d$ be a Lebesgue measurable set and let $f : D \to \mathbb{R}$ be any function. For $r > 0$ we use $(osc_r f) : D \to [0, \infty]$ to denote its "r oscillation":

$$(osc_r f)(x) := \sup_{y \in B_r(x)} f(y) - \inf_{y \in B_r(x)} f(y). \tag{1.2}$$

Of course, $\forall C \in \mathbb{R} \;\; osc_r f = osc_r(f + C)$.

Definition 1.1. *Let μ be some constant c times Lebesgue measure on $\mathbb{R}^d$. For $0 < \alpha \leq 1$ we define the generalized α-Hölder seminorm of f as*

$$|f|_{\alpha;gH} := \sup_{r>0} \frac{1}{r^\alpha} \int_D (osc_r f)(x) \mathrm{d}\mu(x) = c \sup_{r>0} \frac{1}{r^\alpha} \int_D (osc_r f)(x) \mathrm{d}x,$$

where $\mathrm{d}x$ *denotes integration w.r.t. Lebesgue measure. We say that* f *is generalized* α*-Hölder continuous if* $|f|_{\alpha;gH} < \infty$.

It is easy to see that $osc_r f$ is indeed Lebesgue measurable, and the value of $|f|_{\alpha;gH}$ would not change if we used closed balls instead of open ones. The factor c is only included for generality – interesting cases are $c = 1$ and $c = \frac{1}{Leb(D)}$.

Remark 1.2. *This definition coincides with the one given by Chernov in [2]. It is also similar to what Saussol calls the "quasi-Hölder property" in [5] (which is a special case of the notion defined by Keller in [4]). However, it is not exactly the same. The difference is that Keller [4] and Saussol [5] use* essential *supremum and infimum in the definition (1.2) of the oscillation, so their definition does not notice the difference between functions that are equal almost everywhere - w.r.t some distinguished (in our case, Lebesgue) measure. This is in accordance with using absolutely continuous measures only, when integrating* f.

From the point of view of the applications we have in mind, two functions, which are equal μ*-almost everywhere, may be very different. Indeed, in these applications we integrate* f *w.r.t. measures which are singular w.r.t.* μ *– actually, concentrated on submanifolds. So, for us, the notion of oscillation with the true* sup *and* inf *is the good one.*

The main result of this paper is the following theorem.

Theorem 1.3. *For any Lebesgue measurable* $D \subset \mathbb{R}^d$*, any bounded* $f : D \to \mathbb{R}$*, any* $r > 0$ *and any* $0 < \alpha \leq 1$

$$|osc_r f|_{\alpha;gH} \leq 2(\sup_D f - \inf_D f)\mu(Conv(D)) \left(\frac{2d+1}{r}\right)^{\alpha},$$

where $Conv(D)$ *denotes the convex hull of* D.

The direct motivation for this theorem is the paper [1], where it is explicitly applied in an argument about mixing for a dynamical system. However, I believe that the result and the proof are of interest on their own.

In the course of the proof, we need to study "approach" maps on $\mathbb{R}^d$, which take every point the same Δ distance closer to some target set H (provided they are far enough). We need to control the effect of this approach map on Lebesgue measure. This study is done in Section 3. The main result there is Theorem 3.1, which is quite natural, but I could not find it in the literature.

2 Proof of the main theorem

Proof of Theorem 1.3. Without loss of generality, we assume that $c = 1$. Let $\hat{D}$ be the closure of $Conv(D)$, and let us extend f to $\hat{D}$ in an arbitrary way preserving the infimum and supremum. Then the left hand side can only grow, while the right hand side remains unchanged since $Leb(\hat{D}) = Leb(Conv(D))$. So it is enough to show the statement for D convex and closed, which we assume from now on.

We write

$$osc_r f = g_1 - g_2$$

with

$$g_1(x) := g_{1,r}(x) := \sup_{y \in B_r(x)} f(y), \tag{2.1}$$

$$g_2(x) := g_{2,r}(x) := \inf_{y \in B_r(x)} f(y).$$

Clearly $|osc_r f|_{\alpha;gH} \leq |g_1|_{\alpha;gH} + |g_2|_{\alpha;gH}$, so it is enough to show that

$$|g_1|_{\alpha;gH} \leq (\sup_D f - \inf_D f) Leb(D) \left(\frac{2d+1}{r}\right)^{\alpha} \tag{2.2}$$

and

$$|g_2|_{\alpha;gH} \leq (\sup_D f - \inf_D f) Leb(D) \left(\frac{2d+1}{r}\right)^{\alpha}. \tag{2.3}$$

We show (2.2). (Then (2.3) is a trivial consequence substituting $f \to (-f)$.) To show (2.2), we can assume, without loss of generality, that

$$0 \leq f \leq K := \sup_D f - \inf_D f. \tag{2.4}$$

Now we take some $\delta > 0$, and estimate the integral of $osc_\delta g_1$.

If $\delta \geq \frac{r}{2d+1}$, we use the trivial estimate $osc_\delta g_1 \leq K$ to get that

$$\frac{1}{\delta^\alpha} \int_D osc_\delta g_1 \mathrm{d}x \leq \frac{1}{\delta^\alpha} \int_D K \mathrm{d}x \leq \left(\frac{2d+1}{r}\right)^{\alpha} K Leb(D),$$

which is exactly what we need to show.

So from now on, we assume that

$$\delta < \frac{r}{2d+1}, \tag{2.5}$$

implying in particular that $\delta < r$. Using the definition (2.1) of g_1 we can write

$$(osc_\delta g_1)(x) = \sup_{y \in B_\delta(x)} \sup_{z \in B_r(y)} f(z) - \inf_{y \in B_\delta(x)} \sup_{z \in B_r(y)} f(z). \tag{2.6}$$

The first term is simply

$$\sup_{y \in B_\delta(x)} \sup_{z \in B_r(y)} f(z) = \sup_{z \in B_{r+\delta}(x)} f(z).$$

To estimate the second term, notice that for any $y \in B_\delta(x)$, if $|x - z| < r - \delta$, then $|y - z| < r$, so $B_{r-\delta}(x) \subset B_r(y)$, implying that

$$\sup_{z \in B_r(y)} f(z) \geq \sup_{z \in B_{r-\delta}(x)} f(z) \quad \text{for any } y \in B_\delta(x),$$

so

$$\inf_{y \in B_\delta(x)} \sup_{z \in B_r(y)} f(z) \geq \sup_{z \in B_{r-\delta}(x)} f(z).$$

Writing these back to (2.6) we get that

$$osc_\delta g_1 \leq h_1 - h_2 \tag{2.7}$$

with

$$h_1(x) := \sup_{z \in B_{r+\delta}(x)} f(z) \quad , \quad h_2(x) := \sup_{z \in B_{r-\delta}(x)} f(z).$$

These $h_1, h_2 : D \to \mathbb{R}$ are easily seen to be Borel measurable.

We want to estimate $\int_D osc_\delta g_1 \leq \int_D h_1 - \int_D h_2$ from above. The idea is roughly that if some $u \in [0, K]$ is obtained as $u = h_1(x)$ for some $x \in D$, then the same u is also obtained as $u = h_2(\tilde{x})$ for some (possibly other) $\tilde{x} \in D$. Moreover, the set of such $\tilde{x}$ cannot be much smaller (in terms of Lebesgue measure), then the set of the x.

To formalise the argument, let μ_1 and μ_2 be measures on $\mathbb{R}$, which are the push-forwards of Lebesgue measure from D to $\mathbb{R}$ by h_1 and h_2, respectively: for any Borel set $A \subset \mathbb{R}$

$$\mu_1(A) := Leb(h_1^{-1}(A)), \quad ; \quad \mu_2(A) := Leb(h_2^{-1}(A)).$$

Notice that both μ_1 and μ_2 are concentrated on $[0, K]$. So integral substitution gives

$$\int_D h_1(x)\mathrm{d}x = \int_{[0,K]} u\mathrm{d}\mu_1(u) \quad , \quad \int_D h_2(x)\mathrm{d}x = \int_{[0,K]} u\mathrm{d}\mu_2(u). \tag{2.8}$$

The idea above is made precise in the following lemma:

Lemma 2.1. *If $\delta < \frac{r}{2d+1}$, then μ_1 is absolutely continuous w.r.t. μ_2, with density*

$$\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2} \leq C = C(r, \delta, d) := \frac{1}{1 - d\frac{2\delta}{r-\delta}}.$$

We postpone the proof of this lemma, and finish the proof of the theorem using the lemma. The lemma implies

$$\int_{[0,K]} u\mathrm{d}\mu_2(u) \geq \int_{[0,K]} u\frac{1}{C}\mathrm{d}\mu_1(u) = \frac{1}{C}\int_{[0,K]} u\mathrm{d}\mu_1(u),$$

so

$$\begin{aligned}\int_{[0,K]} u\mathrm{d}\mu_1(u) - \int_{[0,K]} u\mathrm{d}\mu_2(u) &\leq \left(1 - \frac{1}{C}\right)\int_{[0,K]} u\mathrm{d}\mu_1(u) \leq \\ &\leq \left(1 - \frac{1}{C}\right) K\mu_1([0, K]) = \left(1 - \frac{1}{C}\right) K Leb(D).\end{aligned}$$

The constant factor is $1 - \frac{1}{C(r,\delta,d)} = d\frac{2\delta}{r-\delta}$. Our assumption (2.5) implies that $r - \delta > \frac{2dr}{2d+1}$, so $\frac{1}{r-\delta} < \frac{2d+1}{2d}\frac{1}{r}$. So $1 - \frac{1}{C(r,\delta,d)} \leq (2d+1)\frac{\delta}{r}$. Writing this back to (2.7) using (2.8) gives

$$\int_D (osc_\delta g_1)(x)\mathrm{d}x \leq (2d+1)\frac{\delta}{r} K Leb(D).$$

Using again the assumption (2.5) we get

$$\begin{aligned}\frac{1}{\delta^\alpha}\int_D (osc_\delta g_1)(x)\mathrm{d}x &\leq \delta^{1-\alpha}\frac{2d+1}{r} K Leb(D) \leq \\ \left(\frac{2d+1}{r}\right)^{\alpha-1}\frac{2d+1}{r} K Leb(D) &= \left(\frac{2d+1}{r}\right)^{\alpha} K Leb(D),\end{aligned}$$

which is again exactly what we need to show. Hence Theorem 1.3. $\square$

We are left to prove Lemma 2.1. We will use the notation $H^{(r)}$ to denote the open r-neighbourhood of $H \subset \mathbb{R}^d$ within D:

$$H^{(r)} := \{z \in D : dist(z, H) < r\}.$$

Proof of Lemma 2.1. For any open interval $I = (a, b) \subset \mathbb{R}$ we need to show that $\mu_1(I) \leq C\mu_2(I)$, which is the same as

$$Leb(h_1^{-1}(I)) \leq C Leb(h_2^{-1}(I)). \tag{2.9}$$

To avoid a trivial case, we assume that $h_1^{-1}(I)$ is non-empty. Let

$$H := f^{-1}(I) \subset D.$$

Now if $x \in h_1^{-1}(I)$, meaning that $\sup_{z \in B_{r+\delta}(x)} f(x) \in I$, then $\exists z \in B_{r+\delta}(x) \cap H$, so $dist(x, H) < r + \delta$. (This also means that since $h_1^{-1}(I)$ is non-empty, H is also non-empty.) Using such an x, we construct two candidate points, one of which is certainly in $h_2^{-1}(I)$.

a.) The first candidate point is x itself. If $dist(x, H) < r - \delta$ happens to hold, then $\exists z \in B_{r-\delta}(x)$ such that $f(z) \in I$, so $h_2(x) \geq f(z) > a$. On the other hand, $h_2(x) \leq h_1(x) < b$, so $h_2(x) \in I$ and so $x \in h_2^{-1}(I)$.

b.) To construct the other candidate point, we define a map T on $\mathbb{R}^d \setminus H^{(2\delta)}$ that "takes points 2δ closer to H". To be precise, for any $x \in \mathbb{R}^d$ with $dist(x, H) \geq 2\delta$, let $\pi(x)$ be the point in $\bar{H}$ which is nearest to x.[1] Now define

$$Tx := x + 2\delta \frac{\pi(x) - x}{|\pi(x) - x|}.$$

Since D was assumed to be closed and convex, if $x \in D$ then $\pi(x) \in D$ and $Tx \in D$. This Tx also satisfies $dist(Tx, H) = dist(x, H) - 2\delta \leq r + \delta - 2\delta = r - \delta$, so again $h_2(Tx) > a$. On the other hand, $B_{r-\delta}(Tx) \subset B_{r+\delta}(x)$, so $h_2(Tx) \leq h_1(x) < b$. We got $h_2(Tx) \in I$, so $Tx \in h_2^{-1}(I)$.

Notice that since $\delta < \frac{r}{2d+1} \leq \frac{r}{3}$ by assumption, either $dist(x, H) < r - \delta$ or $dist(x, H) \geq 2\delta$ certainly holds, so for any $x \in D$ either $x \in h_2^{-1}(I)$ or Tx is well defined and $Tx \in h_2^{-1}(I) \subset D$. To write this concisely, we introduce the operation $\mathcal{T}$ on subsets of $\mathbb{R}^d$ as

$$\mathcal{T}\mathcal{A} := (\mathcal{A} \cap H^{(r-\delta)}) \cup T\mathcal{A},$$

where $T\mathcal{A}$ is meant by just ignoring points of $\mathcal{A}$ where T is undefined. With this notation, we just saw that

$$\mathcal{T}(h_1^{-1}(I)) \subset h_2^{-1}(I),$$

so $Leb(h_2^{-1}(I))$ can be estimated from below as

$$Leb(h_2^{-1}(I)) \geq Leb\left(\mathcal{T}(h_1^{-1}(I))\right).$$

Now (2.9) and thus Lemma 2.1 is an immediate consequence of the following Lemma 2.2. □

Lemma 2.2. *For any Lebesgue measurable $\mathcal{A} \subset H^{(r+\delta)}$*

$$Leb(\mathcal{T}\mathcal{A}) \geq \left(1 - d\frac{2\delta}{r - \delta}\right) Leb(\mathcal{A}).$$

Proof. If $\mathcal{A} \subset H^{(r-\delta)}$, then $\mathcal{A} \subset \mathcal{T}\mathcal{A}$, so the statement is trivial. When this is not the case, we will need to understand the effect of $\mathcal{T}$ very precisely. For this purpose, we cut up $\mathcal{A} \setminus H^{(r-\delta)}$ into disjoint sets A_k, based on the number of iterations of T that we can perform without leaving $\mathcal{A}$. The points that can be reached with such iterations will be treated with careful calculations. For the rest, the trivial estimate suffices.

The proof is based on the properties of the map T studied in Section 3. Strictly speaking we will only use Theorem 3.1 about the limited effect of T on Lebesgue measure. The essence of the

[1] If there is more than one such point, let $\pi(x)$ be any one of them. This causes no problem, because there is only one such point for almost every x. The details are written in the proof of Theorem 3.1 and in Remark 3.6.

understanding is that as long as $dist(A,H) > 2\delta$, the map T is one-to-one on A and TA is not much smaller than A.

First, let

$$K := \left\lfloor \frac{r}{2\delta} - \frac{1}{2} \right\rfloor = \max\{k \in \mathbb{N} \,:\, r - (2k+1)\delta \geq 0\}.$$

With this definition, for any point $x \in H^{(r+\delta)} \setminus H^{(r-\delta)}$, $T^k x$ makes sense for $k = 0, 1, \ldots, K$, and possibly for $k = K+1$, but certainly not for $k = K+2$, because $0 \leq dist(T^K x, H) < 4\delta$. For a set $A \subset H^{(r+\delta)} \setminus H^{(r-\delta)}$, the first $K(+1)$ iterates $A, TA, T^2A, \ldots, T^K A$ are disjoint, and of comparable measure. The next iterate $T^{K+1}A$, even if non-empty, can have arbitrarily small measure, so we don't care if it is empty or not, and we will not make use of it in our estimates. This justifies the following definitions:
For $k = 0, 1, \ldots, K-1$

$$\begin{aligned} A_k &:= \{x \in \mathcal{A} \setminus H^{(r-\delta)} \,: Tx \in \mathcal{A}, T^2 x \in \mathcal{A}, \ldots, T^k x \in \mathcal{A}, \text{ but } T^{k+1}x \notin \mathcal{A}\} \\ \mathcal{A}_k &:= A_k \cup TA_k \cup \cdots \cup T^k A_k. \end{aligned} \tag{2.10}$$

On the other hand, for $k = K$,

$$\begin{aligned} A_K &:= \{x \in \mathcal{A} \setminus H^{(r-\delta)} \,: Tx \in \mathcal{A}, T^2 x \in \mathcal{A}, \ldots, T^K x \in \mathcal{A}\} \\ \mathcal{A}_K &:= A_K \cup TA_K \cup \cdots \cup T^K A_K \cup (T^{K+1}A_K \cap \mathcal{A}). \end{aligned} \tag{2.11}$$

For the rest,

$$\mathcal{A}^* := \mathcal{A} \setminus \bigcup_{k=0}^{K} \mathcal{A}_k.$$

These definitions make sure that

$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_K \cup \mathcal{A}^*$$

is a disjoint union, and more importantly, the union

$$\mathcal{T}\mathcal{A} = \mathcal{T}\mathcal{A}_0 \cup \mathcal{T}\mathcal{A}_1 \cup \cdots \cup \mathcal{T}\mathcal{A}_K \cup \mathcal{T}\mathcal{A}^*$$

is also disjoint. This makes the estimation of $Leb(\mathcal{T}\mathcal{A})$ from below feasible. In fact, $\mathcal{T}\mathcal{A}_k = T\mathcal{A}_k$ for every k, while $\mathcal{T}\mathcal{A}^* = \mathcal{A}^*$.

The lemma follows from the following claim: for every $k = 0, 1, \ldots, K$

$$Leb(T\mathcal{A}_k) \geq \left(1 - d\frac{2\delta}{r-\delta}\right) Leb(\mathcal{A}_k). \tag{2.12}$$

Indeed, using the claim, with the notation $\frac{1}{C} = \left(1 - d\frac{2\delta}{r-\delta}\right) < 1$,

$$\begin{aligned} Leb(\mathcal{T}\mathcal{A}) &= \sum_{k=0}^{K} Leb(\mathcal{T}\mathcal{A}_k) + Leb(\mathcal{T}\mathcal{A}^*) = \\ &= \sum_{k=0}^{K} Leb(T\mathcal{A}_k) + Leb(\mathcal{A}^*) \geq \\ &\geq \sum_{k=0}^{K} \frac{1}{C} Leb(\mathcal{A}_k) + \frac{1}{C} Leb(\mathcal{A}^*) = \\ &= \frac{1}{C} Leb(\mathcal{A}), \end{aligned}$$

which is exactly what we have to prove. So we are left to show the claim (2.12).

The key to the calculation is Theorem 3.1, which says in our case that if $2\delta \le \rho \in \mathbb{R}$ and $X \subset D$ is Lebesgue measurable such that $dist(X, H) \ge \rho$, then

$$Leb(TX) \ge \left(\frac{\rho - 2\delta}{\rho}\right)^{d-1} Leb(X).$$

We use this with $X = T^j A_k$ and $\rho := r - (2j+1)\delta \le d(T^j A_k, H)$ (for $0 \le j < k \le K$), to get

$$\frac{Leb(T^{j+1}A_k)}{Leb(T^j A_k)} \ge \left(\frac{r-(2j+3)\delta}{r-(2j+1)\delta}\right)^{d-1}$$

for all j, which implies by induction that

$$\frac{Leb(T^j A_k)}{Leb(A_k)} \ge \left(\frac{r-(2j+1)\delta}{r-\delta}\right)^{d-1},$$

again for all j. The sets $A_k, TA_k, T^2A_k, \dots, T^kA_k$ are pairwise disjoint, so (2.10) and (2.11) give

$$Leb(\mathcal{A}_k) \ge \sum_{j=0}^{k} \left(\frac{r-(2j+1)\delta}{r-\delta}\right)^{d-1} Leb(A_k). \tag{2.13}$$

Our next goal is to estimate $\frac{Leb(\mathcal{A}_k)-Leb(T\mathcal{A}_k)}{Leb(\mathcal{A}_k)}$ from above by estimating the numerator from above and the denominator from below. We make a fine distinction between the cases $k < K$ and $k = K$.

a.) If $k < K$, meaning that $dist(T^kA_k, H) \ge 2\delta$, then "there is room for a $T^{k+1}A_k$", so

$$T(\mathcal{A}_k) = TA_k \cup T^2A_k \cup \dots \cup T^{k+1}A_k,$$

and $Leb(T^{k+1}A_k) \ge \left(\frac{r-(2k+3)\delta}{r-\delta}\right)^{d-1} Leb(A_k)$. Now

$$\begin{aligned} Leb(\mathcal{A}_k) - Leb(T\mathcal{A}_k) = Leb(\mathcal{A}_k) - Leb(T^{k+1}A_k) \le \quad & (2.14)\\ \le \left[1 - \left(\frac{r-(2k+3)\delta}{r-\delta}\right)^{d-1}\right] Leb(A_k). \quad & (2.15)\end{aligned}$$

We estimate the sum in (2.13) with an integral:

$$\frac{Leb(\mathcal{A}_k)}{Leb(A_k)} \ge \int_0^{k+1} \left(\frac{r-(2t+1)\delta}{r-\delta}\right)^{d-1} \mathrm{d}t = \frac{1}{d}\frac{r-\delta}{2\delta}\left[1 - \left(\frac{r-(2k+3)\delta}{r-\delta}\right)^{d}\right].$$

Putting these together, and using that $0 \le \frac{r-(2k+3)\delta}{r-\delta} < 1$, we get that

$$\frac{Leb(\mathcal{A}_k) - Leb(T\mathcal{A}_k)}{Leb(\mathcal{A}_k)} \le d\frac{2\delta}{r-\delta}\frac{1-\left(\frac{r-(2k+3)\delta}{r-\delta}\right)^{d-1}}{1-\left(\frac{r-(2k+3)\delta}{r-\delta}\right)^{d}} \le d\frac{2\delta}{r-\delta}.$$

b.) If $k = K$, then we use

$$Leb(\mathcal{A}_K) - Leb(T(\mathcal{A}_K)) \le Leb(\mathcal{A}_K)$$

and again an integral to estimate the sum in (2.13) (note the careful choice of the upper integration boundary):

$$\frac{Leb(\mathcal{A}_K)}{Leb(A_K)} \geq \int_0^{\frac{r}{2\delta}-\frac{1}{2}} \left(\frac{r-(2t+1)\delta}{r-\delta}\right)^{d-1} \mathrm{d}t = \frac{1}{d}\frac{r-\delta}{2\delta}.$$

Putting these together, we get that

$$\frac{Leb(\mathcal{A}_K) - Leb(T\mathcal{A}_K)}{Leb(\mathcal{A}_K)} \leq d\frac{2\delta}{r-\delta},$$

just like in the previous case.

It immediately follows that

$$\frac{Leb(T\mathcal{A}_k)}{Leb(\mathcal{A}_k)} \geq 1 - d\frac{2\delta}{r-\delta},$$

which is exactly the claim (2.12). □

3 Approach map and measure

Let $\emptyset \neq H \subset \mathbb{R}^d$ and denote its closure by $\bar{H}$. Let $0 < \Delta \in \mathbb{R}$. We define a map $T_\Delta : \mathbb{R}^d \to \mathbb{R}^d$ that "takes points Δ closer to H" in the following way:

- For any $x \in \mathbb{R}^d$ let $\pi(x)$ be the point in $\bar{H}$ which is closest to x – that is, the point $\pi(x) := y \in H$ where the minimum in $d(x,H) = \min\{d(x,y) \,|\, y \in \bar{H}\}$ is obtained. If there is more than one such y, then let $\pi(x)$ be any of them. So $d(x,\pi(x)) = d(x,H)$.
- Now we define the "approach map" $T_\Delta : \mathbb{R}^d \to \mathbb{R}^d$ as

$$T_\Delta x := \begin{cases} x + \Delta\frac{\pi(x)-x}{|\pi(x)-x|}, & \text{if } d(x,H) > \Delta \\ \pi(x), & \text{if } d(x,H) \leq \Delta. \end{cases}$$

This definition implies that

$$d(T_\Delta x, H) = \begin{cases} d(x,H) - \Delta, & \text{if } d(x,H) > \Delta \\ 0, & \text{if } d(x,H) \leq \Delta. \end{cases}$$

The main result of this section is the following.

Theorem 3.1. *If $\emptyset \neq H \subset \mathbb{R}^d$, $A \subset \mathbb{R}^d$ is Lebesgue measurable and $d(H,A) \geq R \geq \Delta \geq 0$, then*

$$Leb(T_\Delta A) \geq \left(\frac{R-\Delta}{R}\right)^{d-1} Leb(A).$$

We prove this through a few lemmas and propositions. The first statement is about the "infinitesimal" version of this approach map, when Δ is very small. We claim that if two points are far away from H, then such a T_Δ does not bring then much closer to each other:

Lemma 3.2. *Let $\tilde{x}, \tilde{y} \in \mathbb{R}^d$, $d(\tilde{x},H) \geq r$ and $d(\tilde{y},H) \geq r$. Let $\tilde{f}(s) = d(T_s\tilde{x}, T_s\tilde{y})$. Then the derivative of $\tilde{f}$ at 0 can be negative, but not too much:*

$$-\dot{\tilde{f}}(0) \leq \frac{\tilde{f}(0)}{r}.$$

Proof. Let $Y = \tilde{y} - \tilde{x}$, $a = \frac{\pi(\tilde{x})-\tilde{x}}{|\pi(\tilde{x})-\tilde{x}|}$, $b = \frac{\pi(\tilde{y})-\tilde{y}}{|\pi(\tilde{y})-\tilde{y}|}$, $R_1 = |\pi(\tilde{x}) - \tilde{x}|$, $R_2 = |\pi(\tilde{y}) - \tilde{y}|$. So $a^2 = b^2 = 1$, $\pi(\tilde{x}) = \tilde{x} + R_1 a$ and $\pi(\tilde{y}) = \tilde{y} + R_2 b = \tilde{x} + Y + R_2 b$. We use the fact that $\pi(\tilde{x})$ is the nearest point of H to $\tilde{x}$, so in particular $d(\pi(\tilde{y}), \tilde{x}) \geq d(\pi(\tilde{x}), \tilde{x})$. Similarly, $\pi(\tilde{y})$ is the nearest point of H to $\tilde{y}$, so $d(\pi(\tilde{x}), \tilde{y}) \geq d(\pi(\tilde{y}), \tilde{y})$. With the above notation these can be written as $|Y + R_2 b| \geq R_1$ and $|R_1 a - Y| \geq R_2$, which are equivalent to

$$bY \geq \frac{R_1^2 - R_2^2 - Y^2}{2R_2}, \tag{3.1}$$

$$aY \leq \frac{R_1^2 - R_2^2 + Y^2}{2R_1}. \tag{3.2}$$

An explicit calculation gives $\tilde{f}(t) = |\tilde{y} + tb - (\tilde{x} + ta)| = |Y + t(b - a)|$, so $\tilde{f}(0) = |Y|$ and

$$-\dot{\tilde{f}}(0) = \frac{1}{|Y|} Y(a - b) = \frac{Ya - Yb}{|Y|}.$$

This can be estimated from above directly using the assumptions as formulated in (3.1) and (3.2) to give

$$\begin{aligned} -\dot{\tilde{f}}(0) &\leq \frac{1}{|Y|}\left[\frac{R_1^2 - R_2^2 + Y^2}{2R_1} - \frac{R_1^2 - R_2^2 - Y^2}{2R_2}\right] = \\ &= \frac{1}{|Y|}\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(Y^2 - (R_1 - R_2)^2\right). \end{aligned}$$

Using $\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \leq \frac{1}{r}$ and $(R_1 - R_2)^2 \geq 0$ we get

$$-\dot{\tilde{f}}(0) \leq \frac{1}{|Y|}\frac{1}{r} Y^2 = \frac{\tilde{f}(0)}{r}.$$

□

Corollary 3.3. *Let $x, y \in \mathbb{R}^d$, $d(x, H) \geq R$ and $d(y, H) \geq R$. Let $f(t) = d(T_t x, T_t y)$. Then for every $0 \leq t \leq R$*

$$-\dot{f}(t) \leq \frac{f(t)}{R - t}.$$

Proof. Fix some $0 \leq t \leq R$. Let $\tilde{x} = T_t x$, $\tilde{y} = T_t y$ and $r = R - t$. Then $\pi(\tilde{x}) = \pi(x)$, $\pi(\tilde{y}) = \pi(y)$ and the conditions of Lemma 3.2 are satisfied. Moreover, $\tilde{f}(s) = f(t+s)$, so $f(t) = \tilde{f}(0)$ and $\dot{f}(t) = \dot{\tilde{f}}(0)$. Applying the lemma gives exactly the statement of the corollary. □

Proposition 3.4. *If $x, y \in \mathbb{R}^d$, $d(x, H) \geq R$ and $d(y, H) \geq R$, then for any $0 \leq \Delta \leq R$*

$$d(T_\Delta x, T_\Delta y) \geq \frac{R - \Delta}{R} d(x, y).$$

Proof. To avoid a trivial case, assume $d(x, y) \neq 0$. We apply Corollary 3.3. With the function f introduced there, $d(x, y) = f(0)$, $d(T_\Delta x, T_\Delta y) = f(\Delta)$, and the statement of the corollary can be read as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln f(t)) \geq -\frac{1}{R - t}.$$

This implies that

$$\ln \frac{f(\Delta)}{f(0)} = \ln f(\Delta) - \ln f(0) \geq \int_0^\Delta \frac{-1}{R - t}\mathrm{d}t = \ln \frac{R - \Delta}{R}.$$

So
$$\frac{d(T_\Delta x, T_\Delta y)}{d(x,y)} = \frac{f(\Delta)}{f(0)} \geq \frac{R-\Delta}{R}.$$

□

We are interested in the effect of such an approach map on the measure of sets. So for $B \subset \mathbb{R}^d$ and $0 \leq s \leq d$ let $\mathcal{H}^s(B)$ denote the s-dimensional outer Hausdorff measure of B. The next statement is an easy corollary of the previous.

Proposition 3.5. *If $H, A \subset \mathbb{R}^d$, $d(H,A) \geq R \geq \Delta \geq 0$ and $0 \leq s \leq d$, then*
$$\mathcal{H}^s(T_\Delta A) \geq \left(\frac{R-\Delta}{R}\right)^s \mathcal{H}^s(A).$$

Proof. If $\Delta = R$, the statement is trivial. If $\Delta < R$, then the first implication of Proposition 3.4 is that T_Δ is injective, so
$$A = \{T_\Delta^{-1} y \,|\, y \in T_\Delta A\}.$$
As a result, if $\{U_k\}_{k=1}^\infty$ is a covering of $T_\Delta A$, then we can cover A with $\{U_k^-\}_{k=1}^\infty$, where $U_k^- := T_\Delta^{-1}(U_k \cap T_\Delta A)$. Proposition 3.4 implies that
$$diam(U_k) \geq \frac{R-\Delta}{R} diam(U_k^-). \tag{3.3}$$
But by definition, the outer Hausdorff measure is essentially an infimum of $\sum_k diam(U_k)^s$ over coverings $\{U_k\}$:
$$\mathcal{H}^s(A) = \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(A) \tag{3.4}$$
where
$$\mathcal{H}^s_\delta(A) = c_s \inf\left\{\sum_{k=1}^\infty diam(V_k)^s \,|\, diam(V_k) \leq \delta, A \subset \bigcup_{k=1}^\infty V_k\right\}$$
and c_s is some normalising constant. So (3.3) implies that
$$\begin{aligned}
\mathcal{H}^s_\delta(T_\Delta A) &\geq c_s \inf\left\{\sum_{k=1}^\infty \left(\frac{R-\Delta}{R} diam(U_k^-)\right)^s \,|\, diam(U_k) \leq \delta, T_\Delta A \subset \bigcup_{k=1}^\infty U_k\right\} \geq \\
&\geq \left(\frac{R-\Delta}{R}\right)^s c_s \inf\left\{\sum_{k=1}^\infty diam(V_k)^s \,|\, diam(V_k) \leq \frac{R}{R-\Delta}\delta, A \subset \bigcup_{k=1}^\infty V_k\right\} = \\
&= \left(\frac{R-\Delta}{R}\right)^s \mathcal{H}^s_{\frac{R}{R-\Delta}\delta}(A).
\end{aligned}$$
So the definition (3.4) gives the statement of the proposition. □

Applying this proposition with $s = d$ would immediately give a comparison of Lebesgue measures. Our goal, Theorem 3.1 is only a little stronger. We will get it by utilising the fact that Proposition 3.4 is a worst case estimate for the contraction, and there is a direction in which T_Δ does not contract at all.

Proof of Theorem 3.1. We will apply the theory of "area and coarea of Lipschitzian maps" from [3], section 3.2.

Let $f : \mathbb{R}^d \to \mathbb{R}^+$ be defined as $f(x) := d(x, H)$. This f is clearly Lipschitz continuous with Lipschitz constant 1, so it is Lebesgue almost everywhere differentiable. Consider an $x \notin \bar{H}$, so

$f(x) > 0$. If "the point $\pi(x)$ in $\bar{H}$ nearest to x" is not well defined, because there are $y_1 \neq y_2 \in \bar{H}$ such that $d(x, y_1) = d(x, y_2) = d(x, H)$, then the (one-sided) directional derivative of f at x is 1 in both the direction of y_1 and y_2, so f can not be differentiable at x. As a result, this can only happen for a zero Lebesgue measure set of x. On the remaining full measure set of $x \notin \bar{H}$, $\pi(x)$ is well defined, the directional derivative of f is 1 and thus the gradient is the unit vector $\nabla f(x) = \frac{x-\pi(x)}{|x-\pi(x)|}$. In the language of [3], section 3.2, this means that the 1-dimensional Jacobian is $J_1 f = 1$ almost everywhere outside $\bar{H}$.

Theorem 3.2.11 from [3], the "coarea formula" says that if $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitzian, $A \subset \mathbb{R}^m$ is Lebesgue measurable and $m > n$, then

$$\int_A J_n f \mathrm{d}Leb^m = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A \cap f^{-1}\{y\}) \mathrm{d}Leb^n(y).$$

We apply this with $n = d$ and $m = 1$ to the above function $f(x) = d(x, H)$. Since A and $T_\Delta A$ are both disjoint from $\bar{H}$, $J_n f = 1$ almost everywhere on them, and the theorem gives that

$$Leb(A) = \int_0^\infty \mathcal{H}^{d-1}(\{x \in A \,|\, d(x, H) = t\}) \mathrm{d}t \tag{3.5}$$

and similarly for $T_\Delta A$. But

$$\{y \in T_\Delta A \,|\, d(y, H) = t\} = T_\Delta(\{x \in A \,|\, d(x, H) = t + \Delta\}),$$

and Proposition 3.4 implies

$$\mathcal{H}^{d-1}(\{y \in T_\Delta A \,|\, d(y, H) = t\}) \geq \left(\frac{R-\Delta}{R}\right)^{d-1} \mathcal{H}^{d-1}(\{x \in A \,|\, d(x, H) = t + \Delta\}).$$

Writing this back to (3.5) applied with $A \to T_\Delta A$ we get

$$\begin{aligned} Leb(T_\Delta A) &\geq \left(\frac{R-\Delta}{R}\right)^{d-1} \int_0^\infty \mathcal{H}^{d-1}(\{x \in A \,|\, d(x, H) = t + \Delta\}) = \\ &= \left(\frac{R-\Delta}{R}\right)^{d-1} Leb(A). \end{aligned}$$

□

Remark 3.6. [Measurability of $T_\Delta A$]. *On the full measure set of x where $\pi(x)$ is well defined, T_Δ is also well defined. Moreover, by Proposition 3.4 the inverse of T_Δ is Lipschitz continuous and thus Lebesgue measurable. So if $A \subset \mathbb{R}^d$ is Lebesgue measurable, then so is $T_\Delta A$.*

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