Mixing and its rate in 'soft' and 'hard' billiards motivated by the Lorentz process

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Abstract

Billiards corresponding to planar periodic Lorentz processes are considered in the usual (hard) sense and in case the hard core potential of the scatterers is replaced by some other circularly symmetric potential. A review on certain important aspects of the history of the subject is given and some new results on exponential decay of correlations are formulated. Both the results from the literature and those of our own mentioned are mathematically rigorous, nevertheless, proofs are only briefly sketched. On further details see the preprint [3].

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1 Introduction

Consider the motion of a point particle in a planar periodic array of circular scatterers. There are several physically relevant dynamical systems related to such a geometrical configuration. For all of these, motion outside the scatterers is uniform, i.e. the point particle proceeds along a straight (geodesic) line with constant (unit) velocity. If the particle bounces off the circles according to the laws of elastic collision (angle of incidence equals to the angle of reflection), we talk about the *hard Lorentz process*. In case its motion is governed by some axis–symmetric potential $V(r)$ that allows the particle to enter the scatterers (the potential vanishes identically outside), we talk about *soft Lorentz processes*. Following tradition we exploit the periodicity of the configuration and investigate the motion of our point particle on the two dimensional flat torus. We are interested in the mathematically rigorous treatment of hyperbolic, ergodic, and statistical properties of these systems.

In order to formulate results we fix some convention and notation. For technical reasons the dynamics are considered in discrete time, i.e. our phase space M is the union of the boundaries of the scatterers supplied with all possible outgoing velocities of unit length. Notation for the first return map onto this natural Poincaré section of our Hamiltonian flow will be T. Full energy E and consequently the length of the inter-collision velocity $|v|$ are invariants of motion. By fixing $|v| = 1$ (and the mass of the particle as $m = 1$) we have $E = \frac{1}{2}$. We assume all the circular scatterers to have the same radius R. Two more important technical conditions on the configuration are *finite horizon* on the one hand and *lack of corner points* (i.e. lack of overlapping or touching scatterers) on the other. As a consequence, the length of free flight between two potential regions in uniformly bounded both from below and above, by two positive constants $t_{\min}$ and $t_{\max}$, respectively. Later on, in certain cases, we will put some extra conditions on $t_{\min}$ (cf. Theorem 3.2).

M can be described by two angular coordinates. The position coordinate Θ is the angle a fixed radius of the circular scatterer makes with a variable one: this latter is the one at the point where the particle leaves the potential. The velocity coordinate φ is measured as the angle of the outgoing velocity and the normal vector of the scatterer at the point of outcome.[1] We have $\Theta \in [0, 2\pi)$ and $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. By projecting the Liouville measure onto M we get a natural invariant measure: $d\mu = \text{const.} \cos(\varphi) d\Theta d\varphi$.

The dynamical role of the potential can be understood by means of the following important quantity. Consider $\Delta\Theta$, the difference of Θ coordinates at the points of entering and leaving the circle. (See figure 1.) By symmetry reasons $\Delta\Theta$ depends merely on φ. The function $\Delta\Theta(\varphi)$ is often termed as the *rotation function* for the potential (see also [6]).

To discuss hyperbolic behaviour (and consequently, prove ergodic or statistical properties) we need to handle the differential aspects of dynamics. Thus the derivative of the rotation function is to be taken. For brevity of notation we introduce $\kappa(\varphi) = \Delta\Theta'(\varphi)$.

In the rest of the paper we address the following questions:

- Is the dynamical system T hyperbolic? (I.e. are the Lyapunov exponents

[1] Actually, by symmetry of the potential the angle of incidence would be $\pi - \varphi$, but tradition is to measure it in the opposite direction, so it will also be φ.

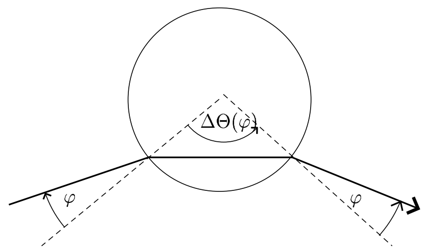


Fig. 1. meaning of the rotation function

nonzero a.e. with respect to μ?)

- Is the dynamical system T ergodic (with respect to μ)?
- Is the rate of mixing – the rate of correlation decay – exponential? (I.e, as we think of correlation decay throughout the paper, given two sufficiently smooth – e.g. Hölder continuous – observables, does their $T-$time correlation function – with respect to μ – necessarily decay exponentially?)

Note that this latter question has crucial significance in physics as correlation functions – and their decay rates – play an important role in the theory of transport phenomena.

As it is shown below, to give definitive answers to the questions phrased above conditions on $\kappa(\varphi)$ (and, in case of mixing rates, on its derivative) are needed. Thus our task is twofold: analyzing dynamics based on κ and obtaining potentials with the suitable κ-behaviour.

The two sections below are organized as follows. In section 2 we give a short historical review on related mathematically rigorous results. In section 3 we point out some recent results of our own on the rate of mixing in soft planar billiards with axis-symmetric potentials. Proofs are only sketched. For a more detailed treatment see [3].

2 Review on the history of the subject

The review below is by no means meant to be complete. We concentrate on mathematically rigorous results with direct relevance to our own research. For a more extensive list of references see the papers [4], [5] and [6].

The mathematical theory of **the hard planar Lorentz process** dates back to the 1970s. It was Ya. Sinai who first obtained hyperbolicity and ergodicity for the corresponding billiard systems in [15]. This paper was the starting point of several generalizations and reformulations, see e.g. [16].

These classical papers – and actually, billiard theory in general – rely on the analysis of *fronts.*[2] A front is a local infinitesimal perturbation of a given trajectory. In case the perturbed and the original trajectory are parallel, do diverge from, or converge to each other, the front is called neutral, convex or concave, respectively. In hard Lorentz processes convex fronts remain convex in positive time and distances on them grow uniformly. Thus they are good 'candidates' for unstable manifolds (actually, unstable manifolds are the convex fronts for which all past iterates are convex fronts as well, see the sketch of proofs in section 3 or more details in [3]).

As to the rate of mixing, in contrast to ergodicity, one had to wait until the late 1990s for the optimal result, exponential decay of correlations. The breakthrough is related to the method of L.S. Young, [17]. In her paper and in N. Chernov's work [4] based on it, exponential mixing has been proven for all major types of planar dispersing billiards (and thus specifically for those corresponding to hard Lorentz processes).

The essence of the papers [17] and [4] is a fine analysis of hyperbolic behaviour, i.e. the growth of unstable manifolds. Besides, the differential aspects of dynamics properties like distortion and curvature bounds are needed. These are related to the inhomogeneities of the derivative. Roughly speaking, in addition to the first, even the second derivative is to be investigated.

As to the case of **soft planar billiards**, we restrict to the case of finite range **axis-symmetric potentials** as natural softenings motivated by Lorentz processes with circular scatterers. Results point into two different directions. On the one hand, for quite general softenings of the potential the chaotic behaviour is no longer present. Stable periodic orbits and islands appear in the phase space. This is generally the case with smooth potentials, see [5] and references therein. Similar phenomena are observed in [13], however, the technique of that paper – corresponding to the presence of tangent periodic orbits in the original billiard – is different from ours and also applies to certain systems beyond the axis-symmetric setting.

In contrast to the above behaviour, in many cases, especially when the potential is not C^1, hyperbolicity and ergodicity persist.[3] Investigation of such soft billiards dates back to the pioneer works of Sinai ([14]) and Kubo et. al. ([9] and [10]). There are two different approaches present in the literature to this hyperbolic case. On the one hand, under conditions on the derivatives (up to the second) of the potential the Hamiltonian flow turned out to be equivalent to a geodesic flow on a negatively curved manifold. This point of

[2] The technically different approach of invariant cones introduced by M. Wojtkowski – see e.g. [11] – exploits, in the case of hyperbolic billiards, similar geometric phenomena.

[3] In [6] there is a smooth potential example with ergodic behaviour, too. However, it is unstable with respect to small perturbations like varying the full energy level.

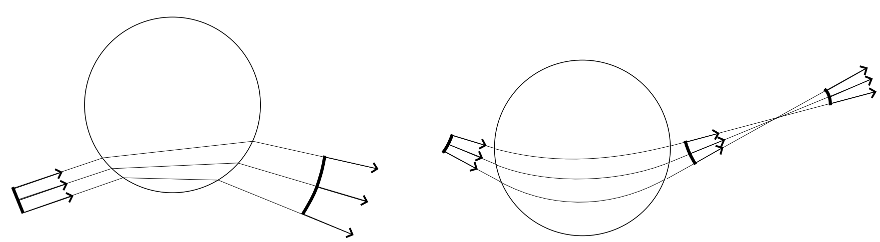
view is especially suitable for potentials with Coulomb type singularities, see [7] on details.

The other approach – which is actually the one we follow, see section 3 – is to study $\kappa(\varphi)$ and its relevance to the evolution of fronts. [12] and especially [6] – which is one of our main references – are written in this spirit.

There is a rather large class of potentials for which [6] obtained ergodicity and hyperbolicity, both repelling and attracting. Reason for their success is that V. Donnay and C. Liverani have found conditions on κ sufficient for hyperbolic behaviour that later on also turned out to be essentially necessary. Before formulating these we turn to a simple, though slightly artificial example. Fix the potential inside the scatterers as identically zero. Thus, when entering the circle, the particle proceeds along its straight trajectory without changing its direction or velocity magnitude. This system is, of course, not hyperbolic: neutral fronts remain neutral and convex ones loose more and more convexity as they evolve. It is straightforward to calculate $\Delta\Theta(\varphi) = \pi - 2\varphi$ and thus $\kappa(\varphi) = -2$ identically.

In view of the above example it is not surprising that the value $\kappa = -2$ is to be avoided. Actually, [6] obtained results in two different cases.

Dispersing case. Assume $\kappa < -2$ or $\kappa \geq 0$ for all φ. The soft billiard is ergodic and hyperbolic. Mechanism of hyperbolicity is, just like in the hard case, related to convex fronts. I.e. incoming (neutral and) convex fronts (those reaching the potential disk) turn into outgoing convex fronts (convex when leaving the disk). See figure 2.



(a) dispersing case

(b) defocusing case

Fig. 2. mechanisms of hyperbolicity

Defocusing case. Assume there is some $\delta > 0$ such that $0 \geq \kappa > -2 + \delta$. In addition, the configuration is such that there is some lower bound $t_{\min}$ on the free flight between two disks (cf. section 1) which satisfies $t_{\min} > \frac{2R(2-\delta)}{\delta}$. The soft billiard is ergodic and hyperbolic. Nevertheless, the mechanism of hyperbolicity is different from the one in the dispersing case. Incoming convex fronts may turn into outgoing concave fronts. These concave fronts, as there is enough time until the next disk is reached, defocus and turn into convex fronts during free flight. Thus, when entering a potential region for the next time, they are convex again. See figure 2.

The above brief discussion sheds some light on the fact why these conditions are, essentially, necessary for chaotic behaviour. Assume there is some φ for which κ approaches -2 from above and the suitable bound on $t_{\min}$ is missing. The outgoing concave fronts do not have enough time to defocus, thus they remain concave for all positive times. In case one can construct a periodic orbit in the vicinity of these persistent concave fronts, the periodic orbit turns out to be stable. [5] shows obstructions for ergodicity roughly along these lines.

3 Recent results on exponential mixing

Our aim in this section is to present briefly some of our recent results on exponential decay of correlations in certain ergodic soft billiards. Based on these results we conjecture that the rate of mixing is exponential in essentially all the (finite horizon) cases for which [6] obtained hyperbolic ergodicity. Nevertheless, in contrast to [6], we do not have many – actually, we do have only two – specific potentials for which exponential decay of correlations can be explicitely shown.

The small number of specific examples is related to the necessity of understanding the 'second derivative of the dynamics' as mentioned above at the beginning of section 2. To obtain curvature and distortion bounds – which is inevitable for the application of methods from [4] and [17] – we need to check the derivative of $\kappa(\varphi)$. More precisely, we introduce the following definition.

Definition 3.1 *The rotation function is termed regular in case the following properties hold.*

(1) $\Delta\Theta(\varphi)$ is piecewise uniformly Hölder continuous. I.e. there are constants $C < \infty$ and $\alpha > 0$, and furthermore, $[-\frac{\pi}{2}, \frac{\pi}{2}]$ can be partitioned into finitely many intervals, such that for any φ_1 and φ_2 (from the interiour of one of the intervals):

$$|\Delta\Theta(\varphi_1) - \Delta\Theta(\varphi_2)| \leq C|\varphi_1 - \varphi_2|^\alpha.$$

(2) $\Delta\Theta(\varphi)$ is a piecewise C^2 function of φ on the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, in the above sense. (Note, however, that κ, in contrast to $\Delta\Theta$, can happen to have no finite one-sided limit at a discontinuity point.)

(3) There is some finite constant C such that

$$|\kappa'(\varphi)| \leq C|\kappa^3(\varphi)|$$

where $\kappa'(\varphi)$ is the derivative of κ with respect to φ.

(4) *For the final property consider any discontinuity point φ_0 where $\kappa(\varphi)$ (in contrast to $\Delta\Theta(\varphi)$) has no finite limit from the left. Of course, in case there is no finite limit from the right, the analogous property is similarly assumed.*
Restricted to some interval $[\varphi_0 - \epsilon, \varphi_0)$; $\omega(\varphi) = \frac{2+\kappa(\varphi)}{\cos\varphi}$ is a monotonic function of φ.

Remark. Note that in case κ is C^1 (or piecewise C^1 with boundedness of itself and of κ') regularity is automatic. In case the asymptotics of κ near some discontinuity is some power law $(\varphi_0 - \varphi)^{-\xi}$ (with $\xi > 0$), regularity means $\frac{1}{2} \leq \xi < 1$.

Now we can formulate our two main theorems. In both of them we assume our general setting, in particular finite horizon and lack of corner points (cf. section 1).

Theorem 3.1 *Assume that*

- *the rotation function is regular,*
- *there is some δ such that $\kappa \geq 0$ or $\kappa < -2 - \delta$ for all φ.*

Correlations (of Hölder-observables, cf. section 1) decay exponentially.

Theorem 3.2 *Assume that*

- *the rotation function is regular,*
- *there is some δ such that $-2 + \delta \leq \kappa < 0$ for all φ,*
- *there is a lower bound $t_{\min} > \frac{2R(2-\delta)}{\delta}$ on the free path.*

Correlations (of Hölder-observables, cf. section 1) decay exponentially.

The proof of these theorems is rather long, thus for brevity we only give a brief sketch of the main argument. The main new difficulty in this problem that we have to overcome is the treatment of quantities connected to the second derivative of the dynamics, especially while traveling through the potential. An analysis finer than before – in this sense – of the evolution of fronts is needed for the proof of curvature bounds, and, especially, distortion bounds (described below). It is also worth mentioning that the choice of the right phase space and metric is not trivial. For example, using the Euclidean metric (as we do) with the phase space of *incoming* particles instead of outgoing, our distortion bounds would no longer hold. For a complete discussion of the proof we refer to [3].

Actually, our theorems are applications of [4]. In this paper N. Chernov showed that given a hyperbolic system with singularities for which certain properties can be shown, correlations decay exponentially. Our proof establishes these properties for the investigated soft billiard systems. More precisely, our theorems are the consequences of Theorem 2.1 from [4] and the properties with a **bold** typeface below. Arguments are to be presented at three different levels.

(1) **Uniform hyperbolicity** *and geometric properties.* We define stable/unstable manifolds as those curves in M that correspond to fronts that remain concave/convex for all future/past iterates of T. Unlike in [6], we need to show that contraction/expansion (in the natural Riemannian metric of M) is indeed uniform. To achieve this, one needs to introduce an auxiliary metric quantity (the p-metric of billiard literature, cf. [3,4]) and furthermore, needs to discuss the cases of $\kappa \geq 0$, $0 > \kappa \geq -1$, $-1 > \kappa > -2+\delta$ and $-2-\delta > \kappa$ separately. In addition, uniform *transversality* of stable and unstable manifolds is to be shown. Besides hyperbolicity, the other main feature of (both hard and soft) billiard dynamics is the presence of *singularities*. These are curves in M corresponding to the images/preimages of tangential collisions and of discontinuities of the rotation function. It is to be shown that singularities and stable/unstable manifolds, even though not necessarily transversal to each other, can have tangential intersections of at most some polynomial rate. This property, often termed as *alignment*, is the point where Hölder continuity of the rotation function (the first property from Definition 3.1) is applied.

(2) *Technicalities on stable/unstable manifolds.* To apply the result of [4] (or more generally, the ideas of L.S. Young from [17]) one needs to show that stable/unstable manifolds enjoy certain regularity properties. The proof of **uniform curvature bounds** relies on the fact that curvature of a convex front (when viewed as a submanifold of the flow phase space) cannot blow up during time evolution as distances on the front always grow faster than the inhomogeneities in its shape. This behaviour is why we need the third regularity property in Definition 3.1. It might be worth mentioning that the proof of curvature bounds seemed much more difficult for the case of $-1 < \kappa < 0$ at first sight as in this case one needs to handle fronts that defocus even within the potential, nevertheless, finally, based on Definition 3.1, we could find an argument that applies to all allowed κ values. As to **distortion bounds** and the **absolute continuity** of holonomy maps, following the idea already applied in 'hard' billiards (cf. [4] and references) homogeneity layers are introduced. Nevertheless, as in addition to the phenomena near tangential singularities common to all billiards further unbounded derivatives at discontinuities of κ may appear, this is to be done with special care. Actually, this is the point where the fourth regularity property from Definition 3.1 is exploited.

(3) **Growth properties of unstable manifolds.** The main idea of the papers [17] and [4] is that expansion of unstable manifolds is uniformly stronger than their fractioning caused by the presence of singularities.

This is quantified by three rather technical growth formulas in [4] (see also [3]). However, establishing the validity of these formulas is the part of [3] which is closest to the analogous discussion from [4]. We note that this is the point where the *alignment* property of singularities and unstable manifolds is used.

Checking all the properties described, the proof of our theorems is complete.

In the two Corollaries below we present two potentials for which calculation of the rotation function is relatively straightforward. In these two cases the assumptions of either Theorem 3.1 or Theorem 3.2 are satisfied. [4]

Corollary 3.3 *Assume the potential equals some positive constant, $V(r) = V_0(\neq 0)$ for all $0 \leq r < R$. If $V_0 < 0$, suppose also that the free path between scatterers is long enough. Correlations decay exponentially. (Actually, we take $V_0 < \frac{1}{2}$ as otherwise the particle could not enter the disk and the system would be equivalent to the corresponding hard billiard.)*

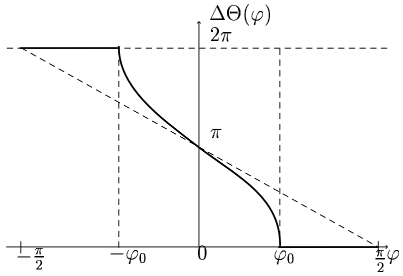
In this constant potential case the equations of motion can be explicitely integrated. Suppose first that $V_0 > 0$. Actually, the problem is equivalent to *diffraction* in geometric optics in case the circles are made of a material which is optically less dense than its environment. For this reason we may introduce $n = \sqrt{1 - 2V_0}$ which plays the role of the relative diffraction coefficient. This time $n < 1$, thus we have the phenomenon of complete reflection for angles greater than φ_0, where $\sin(\varphi_0) = n$. All in all (see also figure 3):

$$\Delta\Theta(\varphi) = \begin{cases} 2 \arccos\left(\frac{\sin(\varphi)}{n}\right) & \text{if } \ |\varphi| < \varphi_0, \\ 0 & \text{if } \ |\varphi| \geq \varphi_0. \end{cases}$$

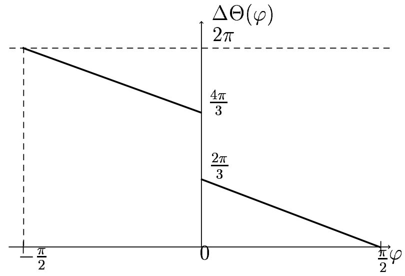
Direct calculation shows that κ is either identically 0 or less than $\frac{-2}{n}$. Regularity can be easily checked, thus the conditions of Theorem 3.1 are satisfied.

It is a natural question what happens when $V_0 < 0$. These attracting potentials correspond to $n > 1$, the case of diffraction with optically dense disks. The rotation function is defined by the same formula (this time there is no need for φ_0 as there is no complete reflection). There is no problem with regularity either. On the other hand, we have $0 > \kappa > \frac{-2}{n}(> -2)$, thus Theorem 3.2 applies in case the free path is suitably bounded from below.

[4] It might be worth mentioning that these potentials, as functions on $\mathbb{R}^2$, are not C^1, thus the equations of motion are to be integrated with care. One needs to integrate inside and outside the disks separately and apply plausible boundary conditions: the magnitude of the velocity at R_- can be obtained from the kinetic energy, and the tangential velocity component is continuous at R.



(a) $V(r) = V_0(> 0)$



(b) $V(r) = \frac{1}{2}\left(1 - \frac{r}{R}\right)$

Fig. 3. rotation function for two examples

Actually, the analysis of this constant potential case from the point of ergodicity dates back to the late eighties, see [1,8,12]. It was shown that for positive V_0 the system is ergodic, and that for negative V_0 its ergodicity depends on the minimum length of the free path. Now we may add that in all ergodic cases the correlations decay exponentially.

Now let us turn to our other example.

Corollary 3.4 *Assume the potential decreases linearly from $\frac{1}{2}$ to 0, i.e.*

$$V(r) = \frac{1}{2}\left(1 - \frac{r}{R}\right).$$

In addition, the free flight is bounded from below: $t_{\min} > 4R$. Correlations decay exponentially.

Although not straightforward either, it is not too difficult to integrate the equations of motion in this case. We have (see also figure 3)

$$\Delta\Theta(\varphi) = \frac{4}{3}\left(\frac{\pi}{2} - \varphi\right)$$

for $\varphi > 0$ and $\Delta\Theta(-\varphi) = -\Delta\Theta(\varphi)$.[5] We have $\kappa = -\frac{4}{3}$ identically, thus Theorem 3.2 applies with $\delta = \frac{2}{3}$.

It is an interesting question what happens when the top of the linearly decreasing potential is different from $\frac{1}{2}$ (the full energy). We cannot integrate the rotation function in an analytic form, nevertheless, we can guess its shape which is, of course, far from linear. In case the 'top' of the potential is less than $\frac{1}{2}$, we expect a smooth rotation function with $\Delta\Theta(0) = \pi$ and $\Delta\Theta(\frac{\pi}{2}) = 0$. Thus there is definitely some φ for which $\kappa = -2$, and the system – at least for

[5] For $\varphi = 0$ the equations of motion do not have a unique solution, nevertheless, this happens on a set of zero measure. The situation is analogous to the usual singularities in billiard theory. On details see [3].

a certain class of configurations, cf. [5] – is most likely not ergodic.[6] In case the 'top' is higher than $\frac{1}{2}$ we expect $\Delta\Theta(0) = 0$ and thus ergodicity is possible in case we have the suitable lower bound on the free path. Nevertheless, the two mechanisms of Theorems 3.1 and 3.2 seem to mix here. These issues are more dealt with in [3].

Needless to say, there are lots of further challenges in this field. It would be important to obtain the rotation function (or merely its relevant properties) for a larger class of potentials, at least numerically. Simulations can play an important role, anyway. For example, once mathematical evidence of the existence of diffusion and further transport coefficients is given, via exponential decay of correlations, one could investigate the dependence of these on certain parameters like the full energy of the particle. Further future perspectives along with more general results and detailed proofs are presented in [3].

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