Logarithmic Sobolev inequality in discrete product spaces:
a proof by a transportation cost distance

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ABSTRACT. Let $\mathcal{X}$ be a finite set and $q^n = \mathcal{L}(X^n)$ a fixed probability measure on $\mathcal{X}^n$. The aim of this talk is to prove an inequality between relative entropy and the sum of average conditional relative entropies of the following form: For any probability measure $p^n = \mathcal{L}(Y^n)$ on $\mathcal{X}^n$

$$D(p^n||q^n) \leq$$

$$Const. \sum_{i=1}^{n} \mathbb{E}_{p^n} D(p_i(\cdot|Y_1,\dots,Y_{i-1},Y_{i+1},\dots,Y_n)||q_i(\cdot|Y_1,\dots,Y_{i-1},Y_{i+1},\dots,Y_n)), \tag{*}$$

where $p_i(\cdot|y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$ and $q_i(\cdot|x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ denote the local specifications for $p^n$ resp. $q^n$, i.e., the conditional distributions of the $i$'th coordinate, given the other coordinates. The constant shall depend on the properties of the local specifications of $q^n$. An inequality (*) can hold only under some weak dependence condition on $q^n$.

(*) directly implies that the Gibbs sampler associated with $q^n$ is a contraction for relative entropy.

Inequality (*) is meaningful in product spaces (both in the discrete and the continuous case), and it can be used to prove a logarithmic Sobolev inequality for $q^n$, provided uniform logarithmic Sobolev inequalities are available for the local specifications of $q^n$.

In the talk I shall try to briefly explain the meaning and significance of logarithmic Sobolev inequalities.

The approach to derive inequality (*), and thereby a logarithmic Sobolev inequality, in discrete product spaces, is to prove contractivity of the Gibbs sampler with respect to an appropriate Wasserstein-like distance.

A logarithmic Sobolev inequality is, roughly speaking, a contractivity property of relative entropy with respect to some Markov semigroup. It is much easier to prove contractivity for a distance between measures, than for relative entropy, since distances satisfy the triangle inequality, and for them well known linear tools, like estimates through matrix norms can be applied.