The real polarization problem

and its connection to products of jointly Gaussian random variables

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Let $x_1, \dots x_n \in \mathbb{R}^n$ be unit vectors, $\|x_j\| = 1$, and consider the following quantity:

$$S = \sup_{\|y\|=1} |\langle x_1, y\rangle \langle x_2, y\rangle \dots \langle x_n, y\rangle|.$$

For which configuration of $x_1, \dots, x_n$ is S minimal? A natural conjecture of Benitez, Sarantopoulos and Tonge states that $S \geq n^{-n/2}$ always holds, where the right hand side corresponds to $x_1, \dots x_n$ being orthonormal. Interestingly, the same conjecture in $\mathbb{C}^n$ has already been solved.

There are several approaches to the problem, analytic [1], geometric [2], and probabilistic [3], yielding partial results. Currently the most promising approach is that of [3] which deduces a lower bound on S from the following:

Theorem If $X_1, \dots X_n$ are jointly Gaussian random variables with zero expectation, then

$$E(X_1^2 \dots X_n^2) \geq EX_1^2 \dots EX_n^2.$$

Equality holds if and only if they are independent or at least one of them is almost surely zero.

A similar result for higher moments would imply the conjecture.

References

[1] Pappas, A.; Revesz, Sz. Gy., Linear polarization constants of Hilbert spaces, J. Math. Anal. Appl. 300 (2004), no. 1, 129–146.

[2] Matolcsi, M., A geometric estimate on the norm of product of functionals, Linear Algebra Appl. 405 (2005), 304–310.

[3] Frenkel, P. E., Pfaffians, hafnians and products of real linear functionals, Math. Res. Lett., to appear.

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