Probability 1
CEU Budapest, fall semester 2018
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Homework sheet 1 – due on 01.10.2018

1.1 **(homework)** Define a σ-algebra as follows:

Definition 1 *For a nonempty set* Ω*, a family* $\mathcal{F}$ *of subsets of* ω *(i.e.* $\mathcal{F} \subset 2^\Omega$*, where* $2^\Omega := \{A : A \subset \Omega\}$ *is the power set of* Ω*) is called a* σ*-algebra over* Ω *if*

- $\emptyset \in \mathcal{F}$
- *if* $A \in \mathcal{F}$*, then* $A^C := \Omega \setminus A \in \mathcal{F}$ *(that is,* $\mathcal{F}$ *is closed under complement taking)*
- *if* $A_1, A_2, \dots \in \mathcal{F}$*, then* $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ *(that is,* $\mathcal{F}$ *is closed under countable union).*

Show from this definition that a σ-algebra is closed under countable intersection, and under finite union and intersection.

1.2 **(homework)** *Continuity of the measure*

a.) Prove the following:

Theorem 1 *(Continuity of the measure)*

i. If $(\Omega, \mathcal{F}, \mu)$ *is a measure space and* $A_1, A_2, \dots$ *is an increasing sequence of measurable sets (i.e.* $A_i \in \mathcal{F}$ *and* $A_i \subset A_{i+1}$ *for all* i*), then* $\mu(\cup_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ *(and both sides of the equation make sense).*

ii. If $(\Omega, \mathcal{F}, \mu)$ *is a measure space,* $A_1, A_2, \dots$ *is a decreasing sequence of measurable sets (i.e.* $A_i \in \mathcal{F}$ *and* $A_i \supset A_{i+1}$ *for all* i*) and* $\mu(A_1) < \infty$*, then* $\mu(\cap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ *(and both sides of the equation make sense).*

b.) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

1.3 **(homework)** Let $\Omega = \{(i,j) \,|\, i,j, \in \mathbb{N}, 1 \le i \le 6\}$ be the set of all 36 possible outcomes in an experiment where we roll a bule and a red die: the result of the experiment is a pair of numbers between 1 and 6, the first number being the number rolled on the blue die, and the second number being the number rolled on the red one.

Let $f : \Omega \to \mathbb{R}$ be given by $f((i,j)) := i + j$, so f is the sum of the two numbers rolled. Clearly, the range of f is $Y := \{2, 3, \dots, 12\}$. Let $\mathcal{G}$ be the discrete σ-algebra on Y and let

$$\mathcal{F} := \{f^{-1}(B) \,|\, B \in \mathcal{G}\},$$

so $\mathcal{F} \subset 2^\Omega$.

a.) Show that $\mathcal{F}$ is a σ-algebra over Ω.

b.) Describe the σ-algebra $\mathcal{F}$: which are the sets that belong to it? Give examples of subsets of Ω that are not in $\mathcal{F}$.