Probability 1
CEU Budapest, fall semester 2016
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Homework sheet 3 – due on 25.10.2016 – and exercises for practice

3.1 *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0,1]$? What is $\lim_{n\to\infty} \left(\int_0^1 f_n(x)dx \right)$ and $\lim_{n\to\infty} \left(\int_0^1 g_n(x)dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples?

(a)
$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write n as $n = 2^k + l$, where $k = 0, 1, 2 \dots$ and $l = 0, 1, \dots, 2^k - 1$ (this can be done in a unique way for every n). Now let
$$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \le x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

3.2 *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:
$$f(x) = \begin{cases} 1 & \text{if } \quad 0 < x,\ 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if } \quad 0 < x,\ 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dx \right) dy$ and $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dy \right) dx$. What's the situation with the Fubini theorem?

3.3 **(homework)** For real numbers $a_1, a_2, a_3, \dots$ define the infinite product $\prod_{k=1}^{\infty} a_k$ as
$$\prod_{k=1}^{\infty} a_k := \lim_{n\to\infty} \prod_{k=1}^{n} a_k,$$
whenever this limit exists.

Let $p_1, p_2, p_3, \dots$ satisfy $0 \le p_k < 1$ for all k. Show that $\prod_{k=1}^{\infty} (1-p_k) > 0$ if and only if $\sum_{k=1}^{\infty} p_k < \infty$.

(Hint: estimate the logarithm of $(1-p)$ with p.)

3.4 Let $X_1, X_2, \dots$ be independent random variables such that
$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \dots X_n}{n} = -1$$

almost surely.

3.5 Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables. Prove that the following two statements are equivalent:

(i) $\mathbb{E}|X_i| < \infty$.

(ii) $\mathbb{P}(|X_n| > n$ for infinitely many n-s$) = 0$.

3.6 **(homework)** Prove that for *any* sequence $X_1, X_2, \dots$ of random variables (real valued, defined on the same probability space) there exists a sequence $c_1, c_2, \dots$ of numbers such that

$$\frac{X_n}{c_n} \to 0 \text{ almost surely.}$$

3.7 Let the random variables $X_1, X_2, \dots, X_n, \dots$ and X be defined on the same probability space. Prove that the following two statements are equivalent:

(i) $X_n \to X$ in probability as $n \to \infty$.

(ii) From every subsequence $\{n_k\}_{k=1}^{\infty}$ a sub-subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ can be chosen such that $X_{n_{k_j}} \to X$ almost surely as $j \to \infty$.

3.8 **(homework)** Let $X_1, X_2, \dots$ be independent such that X_n has *Bernoulli*(p_n) distribution. Determine what property the sequence p_n has to satisfy so that

(a) $X_n \to X$ in probability as $n \to \infty$

(b) $X_n \to X$ almost surely as $n \to \infty$.

3.9 Let $X_1, X_2, \dots$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if there is some $A \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \mathbb{P}(X_n > A) < \infty$.

3.10 Let $X_1, X_2, \dots$ be independent exponentially distributed random variables such that X_n has parameter λ_n. Let $S_n := \sum_{i=1}^{n} X_i$. Show that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$, then $S_n \to \infty$ almost surely, but if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $S_n \to S$ almost surely, where S is some random variable which is almost surely finite.

3.11 Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution *Bernoulli*(p) for some $p \in (0;1)$ but $p \neq \frac{1}{2}$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is absolutely convergent.) Show that the distribution of Y is continuous, but singular w.r.t. Lebesgue measure.

3.12 **(homework)** Let the random variables $X_1, X_2, \dots, X_n, \dots$ and X be defined on the same probability space and suppose that $X_n \to X$ in probability as $n \to \infty$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $Y_n = f(X_n)$ and $Y = f(X)$, show that $Y_n \to Y$ in probability as $n \to \infty$.

(b) Show that if the X_n are almost surely uniformly bounded [that is: there exists a constant $M < \infty$ such that $\mathbb{P}(\forall n \in \mathbb{N}\, |X_n| \le M) = 1$], then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

(c) Show, through an example, that for the previous statement, tha condition of boundedness is needed.

3.13 Let the random variables $X_1, X_2, \ldots, Y_1, Y_2, \ldots, X$ and Y be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

(a) $X_n Y_n \to XY$ in probability.

(b) If almost surely $Y_n \neq 0$ and $Y \neq 0$, then $X_n / Y_n \to X/Y$ in probability.

3.14 **(homework)** Prove that

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \ldots x_n^2}{x_1 + x_2 + \ldots x_n} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n = \frac{2}{3}.$$

3.15 Let $f : [0; 1] \to \mathbb{R}$ be a continuous function. Prove that

(a)

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \ldots x_n}{n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n = f\left(\frac{1}{2}\right).$$

(b)

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \ldots x_n)^{1/n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n = f\left(\frac{1}{e}\right).$$

3.16 Let the random variables $X_1, X_2, \ldots, X_n, \ldots$ be defined on the same probability space and let $Y_n := \sup_{m \geq n} |X_m|$. Prove that the following two statements are equivalent:

(i) $X_n \to 0$ almost surely as $n \to \infty$.

(ii) $Y_n \to 0$ in probability as $n \to \infty$.

3.17 *Weak convergence and densities, again.*

(a) Prove the following

Theorem 1 *Let $\mu_1, \mu_2, \ldots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \ldots$ and f, respectively. Suppose that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \ldots$ and F, respectively. Use the Fatou lemma to show that $F(x) \leq \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.

3.18 **(homework)** Let $X_1, X_2, \ldots$ be independent and uniformly distributed on $[0, 1]$. Let $M_n = \max\{X_1, \ldots, X_n\}$ and let $Y_n = n(1 - M_n)$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*