Probability 1
CEU Budapest, fall semester 2017
Imre Péter Tóth
Homework sheet 4 – due on 07.11.2017 – and exercises for practice

4.1 Let the random variables $X_1, X_2, \dots, X_n, \dots$ and X be defined on the same probability space and suppose that $X_n \to X$ in probability as $n \to \infty$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $Y_n = f(X_n)$ and $Y = f(X)$, show that $Y_n \to Y$ in probability as $n \to \infty$.

(b) Show that if the X_n are almost surely uniformly bounded [that is: there exists a constant $M < \infty$ such that $\mathbb{P}(\forall n \in \mathbb{N}\, |X_n| \le M) = 1$], then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

(c) Show, through an example, that for the previous statement, the condition of boundedness is needed.

4.2 Let the random variables $X_1, X_2, \dots, Y_1, Y_2, \dots, X$ and Y be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

(a) $X_n Y_n \to XY$ in probability.

(b) If almost surely $Y_n \neq 0$ and $Y \neq 0$, then $X_n/Y_n \to X/Y$ in probability.

4.3 Prove that

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 \frac{x_1^2 + x_2^2 + \dots x_n^2}{x_1 + x_2 + \dots x_n} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \dots \mathrm{d}x_n = \frac{2}{3}.$$

4.4 **(homework)** Let $f : [0; 1] \to \mathbb{R}$ be a continuous function. Prove that

(a)

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left(\frac{x_1 + x_2 + \dots x_n}{n}\right) \mathrm{d}x_1 \,\mathrm{d}x_2 \dots \mathrm{d}x_n = f\left(\frac{1}{2}\right).$$

(b)

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left((x_1 x_2 \dots x_n)^{1/n}\right) \mathrm{d}x_1 \,\mathrm{d}x_2 \dots \mathrm{d}x_n = f\left(\frac{1}{e}\right).$$

(Hint: interprete these integrals as expectations.)

4.5 Let the random variables $X_1, X_2, \dots, X_n, \dots$ be defined on the same probability space and let $Y_n := \sup_{m \ge n} |X_m|$. Prove that the following two statements are equivalent:

(i) $X_n \to 0$ almost surely as $n \to \infty$.

(ii) $Y_n \to 0$ in probability as $n \to \infty$.

4.6 *Weak convergence and densities.*

(a) Prove the following

Theorem 1 *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Suppose that $f_n(x) \stackrel{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \dots$ and F, respectively. Use the Fatou lemma to show that $F(x) \le \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.

4.7 Let $X_1, X_2, \dots$ be independent and uniformly distributed on $[0, 1]$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = n(1-M_n)$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*

4.8 **(homework)** Let $X_1, X_2, \dots$ be independent and exponentially distributed with parameter $\lambda = 1$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = M_n - \ln n$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*

4.9 *Poisson approximation of the binomial distribution.* Fix $0 < \lambda \in \mathbb{R}$. Show that if X_n has binomial distribution with parameters (n, p) such that $np \to \lambda$ as $n \to \infty$, then X_n converges to $Poi(\lambda)$ weakly.

4.10 **(homework)** *Continuous limit of the geometric distribution.* Let X_n be geometrically distributed with parameter $p_n = \frac{1}{n}$ and let $Y_n = \frac{1}{n}X_n$. (So $\mathbb{E}Y_n = 1$.) Find the weak limit of Y_n. *(Hint: you can use the method of characteristic functions, but you can also calculate the limiting distribution function directly.)*

4.11 Let X be uniformly distributed on $[-1; 1]$, and set $Y_n = nX$.

a.) Calculate the characteristic function ψ_n of Y_n.

b.) Calculate the pointwise limit $\lim\limits_{n\to\infty} \psi_n(t)$, if it exists.

c.) Does (the distribution of) Y_n have a weak limit?

d.) How come?

4.12 Show that if Ψ is the characteristic function of some random variable X, then the complex conjugate $\bar{\Psi}$ is also the characteristic function of some random variable Y. *(Hint: try to find out what Y is.)*

4.13 Durrett [1], Exercise 3.3.1 *(Hint: try to find the appropriate random variables. Use the previous exercise.)*

4.14 Durrett [1], Exercise 3.3.3

4.15 Durrett [1], Exercise 3.3.9

4.16 Durrett [1], Exercise 3.3.10. Show also that independence is needed.

4.17 Durrett [1], Exercise 3.3.11

4.18 Let $X_1, X_2, \ldots$ be i.i.d. random variables with density (w.r.t. Lebesgue measure) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. (So they have the Cauchy distribution.) Find the weak limit (as $n \to \infty$) of the average

$$\frac{X_1 + \cdots + X_n}{n}.$$

Warning: this is not hard, but also not as trivial as it may seem. Hint: a possible solution is using characteristic functions. Calculating the characteristic function of the Cauchy distribution is a little tricky, but you can look it up.

References

[1] Durrett, R. *Probability: Theory and Examples.* Cambridge University Press (2010)