Probability 1
CEU Budapest, fall semester 2018
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Homework sheet 4 – due on 12.11.2018

4.1 Show that if $X_n \Rightarrow X$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(X_n) \Rightarrow f(X)$.

4.2 Let $F : \mathbb{R} \to [0, 1]$ be a probability distribution function, and let Y be a random variable which is uniformly distributed in $[0, 1]$. Let $X = \sup\{x|F(x) < Y\}$. Show that the distribution function of X is exactly F.

4.3 **(homework)** For a distribution function $F : \mathbb{R} \to [0, 1]$, define its generalized inverse $F^{-1} : [0, 1] \to \bar{\mathbb{R}}$ as $F^{-1}(y) := \sup\{x \in \mathbb{R}|F(x) < y\}$. Let $F, F_1, F_2, \dots$ be distribution functions such that $F_n \Rightarrow F$. Let $\Omega = [0, 1]$, let $\mathbb{P}$ be Lebesgue measure on Ω, and define de random variables $X(\omega) := F^{-1}(\omega)$, $X_n(\omega) := F_n^{-1}(\omega)$ for $\omega \in \Omega$. Show that $X_n \to X$ almost surely.

4.4 **(homework)** Durrett [1], Exercise 3.2.6

4.5 Durrett [1], Exercise 3.2.9

4.6 Durrett [1], Exercise 3.2.12

4.7 Durrett [1], Exercise 3.2.14

4.8 Durrett [1], Exercise 3.2.15

4.9 **(homework)** *Weak convergence and densities.*

(a) Prove the following

Theorem 1 *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Suppose that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \dots$ and F, respectively. Use the Fatou lemma to show that $F(x) \le \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.

4.10 **(homework)** Let $X_1, X_2, \dots$ be independent and uniformly distributed on $[0, 1]$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = n(1 - M_n)$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*

4.11 Let $X_1, X_2, \dots$ be independent and exponentially distributed with parameter $\lambda = 1$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = M_n - \ln n$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*

4.12 Let $S = \mathbb{Z}$ and let the random variables $X, X_1, X_2, \dots \in S$.

a.) Show that $X_n \Rightarrow X$ if and only if $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$ as $n \to \infty$ for every $k \in S$.

b.) It this also true for some arbitrary countable $S \subset \mathbb{R}$?

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)