Probability 1
CEU Budapest, fall semester 2016
Imre Péter Tóth
Homework sheet 6 – due on 21.11.2013 – and exercises for practice

6.1 Durrett [1], Exercise 5.2.1

6.2 **(homework)** Durrett [1], Exercise 5.2.3

6.3 **(homework)** Durrett [1], Exercise 5.2.4

6.4 Let $0 \le p \le 1$ and $q = 1-p$. Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.) Show that $M_n := S_n - n(p-q)$ is a martingale (w.r.t. the natural filtration).

6.5 **(homework)** Let $0 \le p \le 1$ and $q = 1-p$. Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.)

a.) Show that $M_n := \left(\frac{q}{p}\right)^{S_n}$ is a martingale (w.r.t. the natural filtration).

b.) Let $H \subset \mathbb{N}$ and let τ be the random time when the random walk first reaches H, so

$$\tau = \inf\{n \mid S_n \in H\}.$$

Show that $M_{\tau \wedge n}$ is also a martingale.

6.6 **SORRY, the first version of this exercise was totally wrong!** Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple symmetric random walk starting from $S_0 = 0$. Show that $S_n^2 - n$ is a martingale (w.r.t. the natural filtration). *This is a special case of Durrett [1], Exercise 5.2.6. You can also solve that – it' not any harder.*

6.7 **(homework)** *(Pólya's urn)* In an urn there is initially (at time $n = 0$) a black and a white ball. At each time step $n = 1, 2, \dots$

- we draw a ball from the urn, uniformly at random,
- we look at its colour,
- we put it back, and we add another ball of the same colour.

(So we add exactly one ball in each step.) Let X_n be the number of white balls in the urn after n steps, and let $M_n = \frac{X_n}{n+2}$ be the proportion of white balls after n steps.

a.) Show that X_n is uniform on $\{1, 2, \dots, n+1\}$. *(Hint: a possible solution is by induction.)*

b.) Show that M_n is almost surely convergent.

c.) What is the distribution of $M_\infty := \lim_{n\to\infty} M_n$?

6.8 Durrett [1], Exercise 5.2.7

6.9 Durrett [1], Exercise 5.2.9

6.10 Durrett [1], Exercise 5.2.13

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)