Probability 1
CEU Budapest, fall semester 2018
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Homework sheet 6 – due on 3.12.2018

6.1 Consider the probability space $\Omega = \{a, b, c\}$ equipped with the uniform measure as $\mathbb{P}$ (so $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \mathbb{P}(\{c\}) = \frac{1}{3}$). Let the random variable $X : \Omega \to \mathbb{R}$ be such that $X(a) = X(b) = 0$, $X(c) = 1$.

a.) Let D_1 be the partition $\{\{a\}, \{b, c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_1)$ (which is the same as $\mathbb{E}(X|G_1)$, where the σ-algebra G_1 is $G_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$.)

b.) Let D_2 be the partition $\{\{a, b\}, \{c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_2)$ (which is the same as $\mathbb{E}(X|G_2)$, where the σ-algebra G_2 is $G_2 = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$.)

6.2 **(homework)** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\Omega = [0, 1] \times [0, 1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}$ is the Lebesgue measure on ω (restricted to $\mathcal{F}$). Let $\mathcal{G}$ be the σ-algebra

$$\mathcal{G} = \{B \times [0, 1] \mid B \subset [0, 1] \text{ is a Borel set}\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(x, y) = x(x + y)$. Calculate $\mathbb{E}(X|\mathcal{G})$.

6.3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\Omega = [0, 1] \times [0, 1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}$ is the Lebesgue measure on ω (restricted to $\mathcal{F}$). Let $\mathcal{G}$ be the σ-algebra

$$\mathcal{G} = \{[0, 1] \times B \mid B \subset [0, 1] \text{ is a Borel set}\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(x, y) = x^2 + y^2$. Calculate $\mathbb{E}(X|\mathcal{G})$.

6.4 Let ξ and η be independent random variables uniformly distributed on $(0, 1)$. Let $X = \xi\eta$ and $Y = \xi/\eta$. Calcualte $\mathbb{E}(X|Y)$.

6.5 Durrett [1], Exercise 5.1.1

6.6 Durrett [1], Exercise 5.1.3

6.7 Durrett [1], Exercise 5.1.4

6.8 **(homework)** Durrett [1], Exercise 5.1.6

6.9 Durrett [1], Exercise 5.2.1

6.10 Durrett [1], Exercise 5.2.3

6.11 Durrett [1], Exercise 5.2.4

6.12 **(homework)** Let X_n be a martingale w.r.t. the filtration $\mathcal{F}_n$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the random variable $\tau : \Omega \to \mathbb{N}$ be a *stopping time*, meaning

$$\{\tau = k\} := \{\omega \in \Omega \mid \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for every } k.$$

Using the notation $a \wedge b := \min\{a, b\}$, we introduce the process

$$Y_n := X_{\tau \wedge n} = \begin{cases} X_n & \text{if } n < \tau, \\ X_\tau & \text{if } n \geq \tau. \end{cases}$$

Show that Y_n is also a martingale w.r.t. $\mathcal{F}_n$. (Hint: Y_n is the fortune of a gambler with a certain strategy.)

6.13 **(homework)** Let $a, b \in \mathbb{Z}$ with $a < 0 < b$. Let S_n be a simple symmetric random walk with $S_0 = 0$ and let τ be the first hitting time for $\{a, b\}$. Apply the optional stopping theorem to the martingale S_n to find the hitting probabilities $p_a = \mathbb{P}(S_\tau = a)$ and $p_b = \mathbb{P}(S_\tau = b)$.

6.14 Let $p \in (0, 1)$ be fixed, and let $q = 1 - p$. A frog performs a (discrete time) random walk on the 1-dimensional lattice $\mathbb{Z}$ the following way:

The initial position is $X_0 = 0$. The frog jumps 1 step up with probability p and jumps 1 step down with probability q at each time step, independently of what happened before, until it reaches either the point $a = -10$ or the point $b = +30$, which are *sticky*: if the frog reaches one of them, it stays there forever.

Let X_n denote the position of the frog after n steps (for $n = 0, 1, 2, \dots$).

a.) Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is a martingale (w.r.t. the natural filtration).

b.) Show that Y_n converges almost surely to some limiting random variable Y_∞. What are the possible values of Y_∞?

c.) How much is $\mathbb{E}Y_\infty$ and why?

d.) Suppose now that $p \neq \frac{1}{2}$. Use the previous results to calculate the probability that the frog eventually gets stuck at the point $a = -10$.

6.15 Let $0 \leq p \leq 1$ and $q = 1 - p$. Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.) Show that $M_n := S_n - n(p - q)$ is a martingale (w.r.t. the natural filtration).

For $p \neq q$, use this to find the expectation of the time when the frog of Exercise 14 gets stuck.

6.16 Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple symmetric random walk starting from $S_0 = 0$.

a.) Show that $S_n^2 - n$ is a martingale (w.r.t. the natural filtration). *This is a special case of Durrett [1], Exercise 5.2.6. You can also solve that – it' not any harder.*

b.) Use this and the result of Exercise 13 to find the expectation of the stopping time when the walk first reaches either -10 or 30.

c.) How about the expectation of the stopping time when the walk first reaches 30?

6.17 Let $\mathcal{F}_n$ be a filtration and X any random varibale with $\mathbb{E}|X| < \infty$. Let $X_n = \mathbb{E}(X|\mathcal{F}_n)$.

a.) Show that X_n is a martingale w.r.t. $\mathcal{F}_n$.

b.) Show that X_n converges almost surely to some limit X_∞.

c.) Give a specific example when $X_\infty \neq X$.

d.) Give a specific example when $X_\infty = X$.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)