Probability 1
CEU Budapest, fall semester 2018
Imre Péter Tóth
Final exam, 10.12.2018
Working time: 150 minutes
Every question is worth 10 points. Maximum total score: 40 points.

1. Calculate the limit
$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^1 \int_0^1 \dots \int_0^1 \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}\, \mathrm{d}x_1\, \mathrm{d}x_2 \dots \mathrm{d}x_n.$$
Justify your answer.

2. State the continuity theorem for characteristic functions. State the central limit theorem (CLT). Sketch the proof of the CLT using the continuity theorem.

3. a.) Show that if X_n is a martingale adapted to the filtration $\mathcal{F}_n$, then X_n^2 is a submartingale (adapted to the same filtration).

   b.) Show an example of a submartingale Y_n (adapted to a filtration $\mathcal{F}_n$) such that Y_n^2 is not a submartingale.

4. A frog keeps jumping on the set of nonnegative integer numbers, in discrete time $n = 0, 1, 2, \dots$ with the following rule. At time $n = 0$ it is at $X_0 = 1$. For all $n \geq 1$, at time n, if it is at $k \geq 0$, then it

   - jumps to $k+1$ with probabilty $\frac{k}{2(k+1)n}$,
   - jumps to $k-1$ with probabilty $\frac{k}{2(k+1)n}$,
   - stays at k (does not jump) with the remaining probability $1 - 2\frac{k}{2(k+1)n}$,

   independently of what happened before. (In particular, if the frog is once at 0, then it stays there for ever.) Calculate the probability that the frog reaches 0.

5. A flea performs a simple *asymmetric, trapped* random walk on the set $\{-10, -9, \dots, 9, 10\}$, meaning that in every step

   - it jumps 1 unit "down" with probability $\frac{2}{3}$ and "up" with probability $\frac{1}{3}$, independently of the past, unless it is at one of the endpoints,
   - if it is at the endpoint -10 or 10, then it stays there (forever).

   The flea starts from 0. Show that sooner or later it will reach one of the endpoints, and calculate the probability that this endpoint is 10.

   Hint: One possible solution is to notice (and show) that if the position of the flea after n jumps is X_n, then 2^{X_n} is a martingale.