Probability 1
CEU Budapest, fall semester 2018
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Midterm exam, 31.10.2018
Working time: 120 minutes $\approx \infty$
Every question is worth 10 points. Maximum total score: 30.

1. Do there exist random variables $Y, X_1, X_2, \dots$ such that $X_n \to Y$ almost surely and

   a.) $\lim_{n\to\infty} \mathbb{E}X_n^2 \neq \mathbb{E}Y^2$?

   b.) $\lim_{n\to\infty} \mathbb{E}\frac{1}{1+X_n^2} \neq \mathbb{E}\frac{1}{1+Y^2}$?

   c.) $\lim_{n\to\infty} \mathbb{E}(\mathrm{sign}(X_n)) \neq \mathbb{E}(\mathrm{sign}(Y))$?

   (All the expectations and limits should exist. sign denotes the sign function: $\mathrm{sign}(x) = 1$, if $x > 0$; $\mathrm{sign}(x) = 0$, if $x = 0$; $\mathrm{sign}(x) = -1$, if $x < 0$.)

   If not, why not? If yes, give an explicit example (meaning: a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and functions $Y, X_1, X_2, \dots : \Omega \to \mathbb{R}$)!

2. A frog, which is getting tired, moves along a line in the following way: At time $n = 0$ it starts from the origin. In the nth step (for $n = 1, 2, \dots$), with probability $\frac{1}{2n}$ it jumps length $\frac{1}{n}$ to the left; with probability $\frac{1}{2n}$ it jumps length $\frac{2}{n}$ to the right, and with the remaining probability $1 - \frac{1}{n}$ it stays where it was. Let Z be the position of the frog after infinitely many jumps.

   a.) What is $\mathbb{E}Z$?

   b.) Does Z make sense?

3. Let $p_1, p_2, \dots \in [0, 1]$, let $X_1, X_2, \dots$ be independent random variables with $X_n \sim B(p_n)$ and let $Y_n = \frac{1}{p_n} X_n$. Let
$$Z = \lim_{n\to\infty} \frac{Y_1 + Y_2 + \dots + Y_n}{n}$$
   (almost sure limit). How much is $\mathbb{E}Z$? (Does the limit exist?)

   a.) When $p_n = 2^{-n}$?

   b.) When $p_n = \frac{1}{n}$?

4. We take a big empty bag.

   - 1 minute before midnight, we put 10 balls into the bag (numbered $1, 2, \dots, 10$). Then we draw a ball from the bag at random, and throw it away.
   - $\frac{1}{2}$ minute before midnight, we put 10 more balls into the bag (numbered $11, 12, \dots, 20$). Then we draw a ball from the bag at random, and throw it away.
   - $\frac{1}{4}$ minute before midnight, we put 10 more balls into the bag (numbered $21, 22, \dots, 30$). Then we draw a ball from the bag at random, and throw it away.
   - ... and so on:
   - $\frac{1}{2^n}$ minute before midnight, we put 10 more balls into the bag (numbered $10n+1, 10n+2, \dots, 10n+10$). Then we draw a ball from the bag at random, and throw it away.

   Let X be the number of balls in the bag at midnight – that is:
$$X = \#\{k \in \mathbb{N} \mid \text{ball number } k \text{ is in the bag at midnight}\}.$$
   What is $\mathbb{E}X$?