Probability 1
CEU Budapest, fall semester 2016
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Midterm exam, 25.10.2013 – solutions
Working time: 120 minutes $\approx \infty$
Every question is worth 10 points. Maximum total score: 30.

1. Is there a sequence X_n of random variables on the same probability space such that

   a.) $X_n \to 0$ almost surely, and $\mathbb{E}X_n^2 \to \frac{1}{2}$?
   b.) $X_n \to 0$ almost surely, and $\mathbb{E}\sin(X_n) \to \frac{1}{2}$?

   If no, why not? If yes, give an example!

   Solution:

   a.) Yes. For example, let the probability space be $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1), Borel, Leb)$ and let $X_n : \Omega \to \mathbb{R}$ be
   $$X_n(\omega) = \begin{cases} \sqrt{\frac{n}{2}} & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{if not} \end{cases}.$$
   Then $X_n \to 0$ for every $\omega \in (0,1) = \Omega$, but
   $$\mathbb{E}X_n^2 = \int_\Omega X_n^2 \,\mathrm{d}\mathbb{P} = \int_{(0,1)} X_n^2(\omega)\,\mathrm{d}\omega = \int_0^{\frac{1}{n}} \frac{n}{2}\,\mathrm{d}\omega = \frac{1}{2}$$
   for every n, so $\mathbb{E}X_n^2 \to \frac{1}{2}$.

   b.) No. $f(x) = \sin x$ is continuous and $-1 \le f \le 1$, so if $X_n \to 0$ almost surely, then $Y_n := f(X_n) \to Y := f(0) = 0$ almost surely, and the dominated convergence theorem (or the bounded convergence theorem) ensures that $\mathbb{E}Y_n = \int_\Omega Y_n \,\mathrm{d}\mathbb{P} \to \int_\Omega Y \,\mathrm{d}\mathbb{P} = \mathbb{E}Y = 0$. *(Alternative proof: If $X_n \to 0$ almost surely, then also $X_n \Rightarrow 0$ weakly, so $\mathbb{E}f(X_n) \to \mathbb{E}f(0) = 0$ for the bounded and continuous test function $f(x) = \sin x$.)*

2. Let $X_1, X_2, \dots$ be independent, $X_n \sim B(p_n)$ with $p_n \in [0,1]$. Let $Y = \sum_{n=1}^\infty X_n$.

   a.) Show that if $\sum_{n=1}^\infty p_n < \infty$, then $Y < \infty$ almost surely.
   b.) Show that if $\sum_{n=1}^\infty p_n = \infty$, then $Y = \infty$ almost surely.

   Solution: Let $A_n = \{X_n = 1\}$. So $\mathbb{P}(A_n) = p_n$, and $Y = \infty$ if and only if A_n occurs for infinitely many values of n.

   a.) If $\sum_{n=1}^\infty p_n < \infty$, then this has probability 0 by the first Borel-Cantelli lemma.
   b.) If $\sum_{n=1}^\infty p_n = \infty$, then this has probability 1 by the first Borel-Cantelli lemma, since the A_n are independent, because the X_n are independent.

3. Let $X_1, X_2, \dots$ be random variables on the same probability space, $X_n \sim Exp(\lambda_n)$ with $\lambda_n > 0$. Show that if $\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$, then $\sum_{n=1}^\infty X_n < \infty$ almost surely.

   Solution: Let $Y_N = \sum_{n=1}^N X_n$. Since $X_n \ge 0$, the sequence Y_N is nonnegative and increasing, $Y := \sum_{n=1}^\infty X_n = \lim_{N\to\infty} Y_N$ exists (but it's possibly infinite). The expectations are $\mathbb{E}X_n = \frac{1}{\lambda_n}$ and so $\mathbb{E}Y_N = \sum_{n=1}^N \frac{1}{\lambda_n}$. The monotone convergence theorem ensures that
   $$\mathbb{E}Y = \mathbb{E}\lim_{N\to\infty} Y_N = \lim_{N\to\infty} \mathbb{E}Y_N = \sum_{n=1}^\infty \frac{1}{\lambda_n},$$
   which is finite by assumption. But of course, if $\mathbb{E}Y < \infty$, then $Y < \infty$ almost surely.

4. A kind of molecule is trying to decompose into atoms the following way: At each time $t \in \{\delta, 2\delta, 3\delta, \dots\}$ it tries to decompose, and it always succeeds with probabilty δ, which is very small – if it has not succeeded before. If it fails, it tries again next time, indepenedently of the past attempts. (We measure time in hours).

   Let T_δ denote the random time when it successfully decomposes.

   Find the weak limit of T_δ as $\delta \to 0$. (Find means: describe in your favourite way, or write down its name.)

   Solution: Let $X_\delta = \frac{T_\delta}{\delta} \in \mathbb{N}$, so X_δ is the number of attempts needed to successfully decompose. Clearly X_δ has geometrical distribution with parameter δ, which means that $\mathbb{P}(X_\delta = k) = (1-\delta)^{k-1}\delta$ for $k = 1, 2, \dots$. However, it is more fortunate to look at the tail of the distribution:

$$\mathbb{P}(X_\delta > k) = \mathbb{P}(\text{the first } k \text{ attempts fail}) = (1-\delta)^k \quad \text{for } k = 0, 1, 2, \dots.$$

   For possibly noninteger x this means

$$\mathbb{P}(X_\delta > x) = \mathbb{P}(X_\delta > \lfloor x \rfloor) = (1-\delta)^{\lfloor x \rfloor} \quad \text{for } x \geq 0,$$

   where $\lfloor x \rfloor$ means the lower integer part of x.

   So the distribution function of X_δ is

$$F_{X_\delta}(x) = \mathbb{P}(X_\delta \leq x) = 1 - \mathbb{P}(X_\delta > x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1-\delta)^{\lfloor x \rfloor} & \text{if } x \geq 0 \end{cases}.$$

   Since $T_\delta = \delta X_\delta$, its distribution function is

$$\begin{aligned} F_{T_\delta}(t) &= \mathbb{P}(T_\delta \leq t) = \mathbb{P}(\delta X_\delta \leq t) = \mathbb{P}\left(X_\delta \leq \frac{t}{\delta}\right) = \\ &= F_{X_\delta}\left(\frac{t}{\delta}\right) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - (1-\delta)^{\lfloor \frac{t}{\delta} \rfloor} & \text{if } t \geq 0 \end{cases}. \end{aligned}$$

   By elementary calculus, for $t > 0$

$$\lim_{\delta \to 0} F_{T_\delta}(t) = \lim_{\delta \to 0} \left[1 - (1-\delta)^{\lfloor \frac{t}{\delta} \rfloor}\right] = 1 - \exp\left(\lim_{\delta \to 0}\left[-\delta \left\lfloor \frac{t}{\delta} \right\rfloor\right]\right) = 1 - e^{-t},$$

   so for every $t \in \mathbb{R}$

$$F_{T_\delta}(t) \to F(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-t} & \text{if } t \geq 0 \end{cases}$$

   as $\delta \to 0$. So we have shown $F_{T_\delta} \Rightarrow F$ where F is the distribution function of the exponential distribution with parameter 1, so $T_\delta \overset{\delta \to 0}{\Rightarrow} Exp(1)$.